University of Bahrain

Department of Electrical and Electronics Engineering

EENG372

Communication Systems I

Dr. Sana Almansoori

Topic 2:

Frequency Modulation (FM)

This Topic will cover

- Frequency Modulation (FM)
- FM Generation
- FM Demodulation
- FM Receivers

Modulation

Q: What are the different types of Modulation?

The carrier is usually a sinusoidal signal:

$$v_c(t) = V_c \cos(2\pi f_c t) = V_c \cos(\theta_c)$$

Three things can be changed by the information signal:

1. Amplitude
2. Angle
   i. Frequency
   ii. Phase

Frequency Modulation

Q: What is Frequency Modulation ?

Angle Modulation is having the message signal alter the frequency of a carrier signal for transmission.

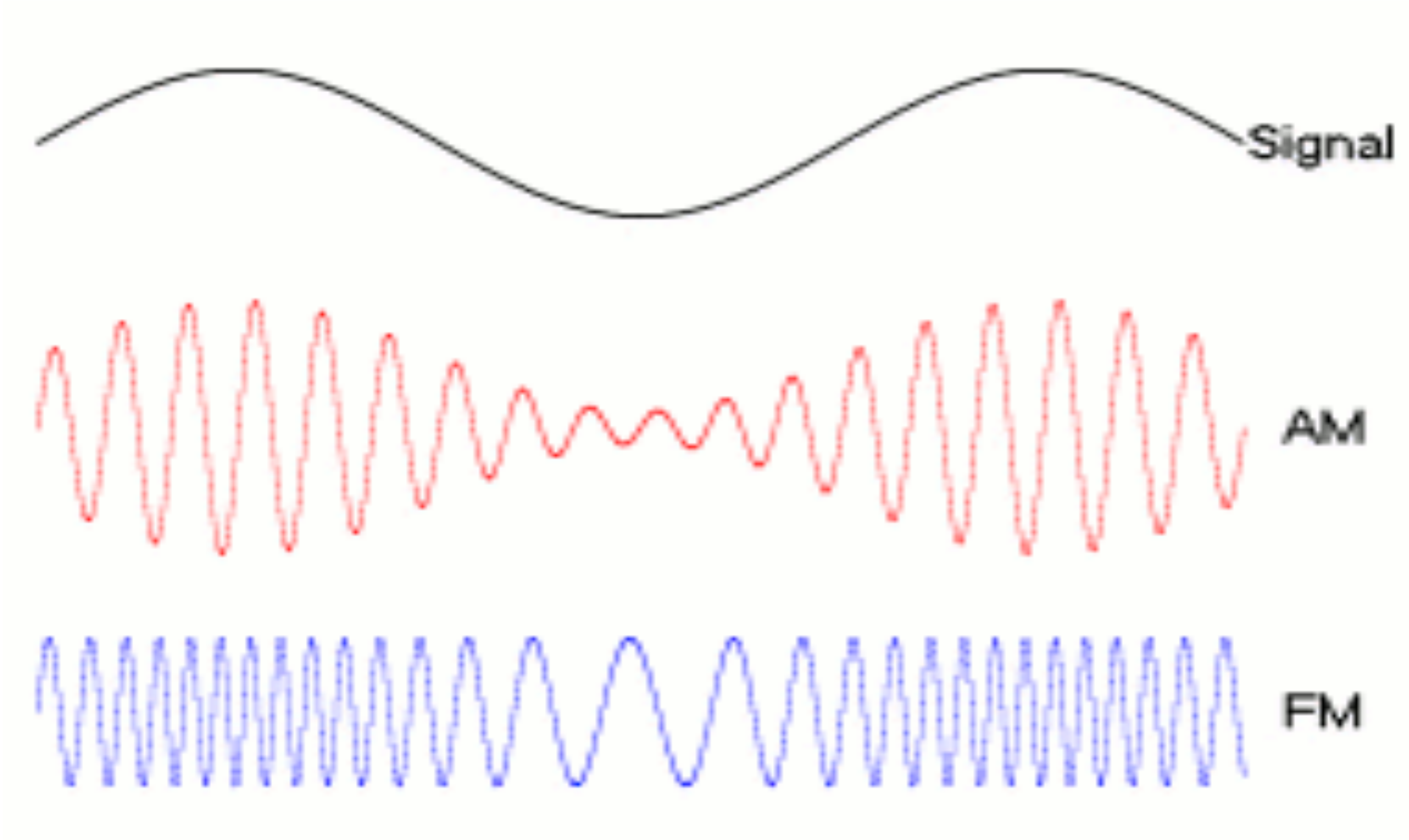
$$v_c(t) = V_c \cos(2\pi f_c t) = V_c \cos(\theta_c)$$

This will lead to the phase of the carrier changing with time (instantaneous phase):

$$v_{ang\_mod}(t) = v_{FM/PM}(t) = V_c \cos(\theta_i(t))$$

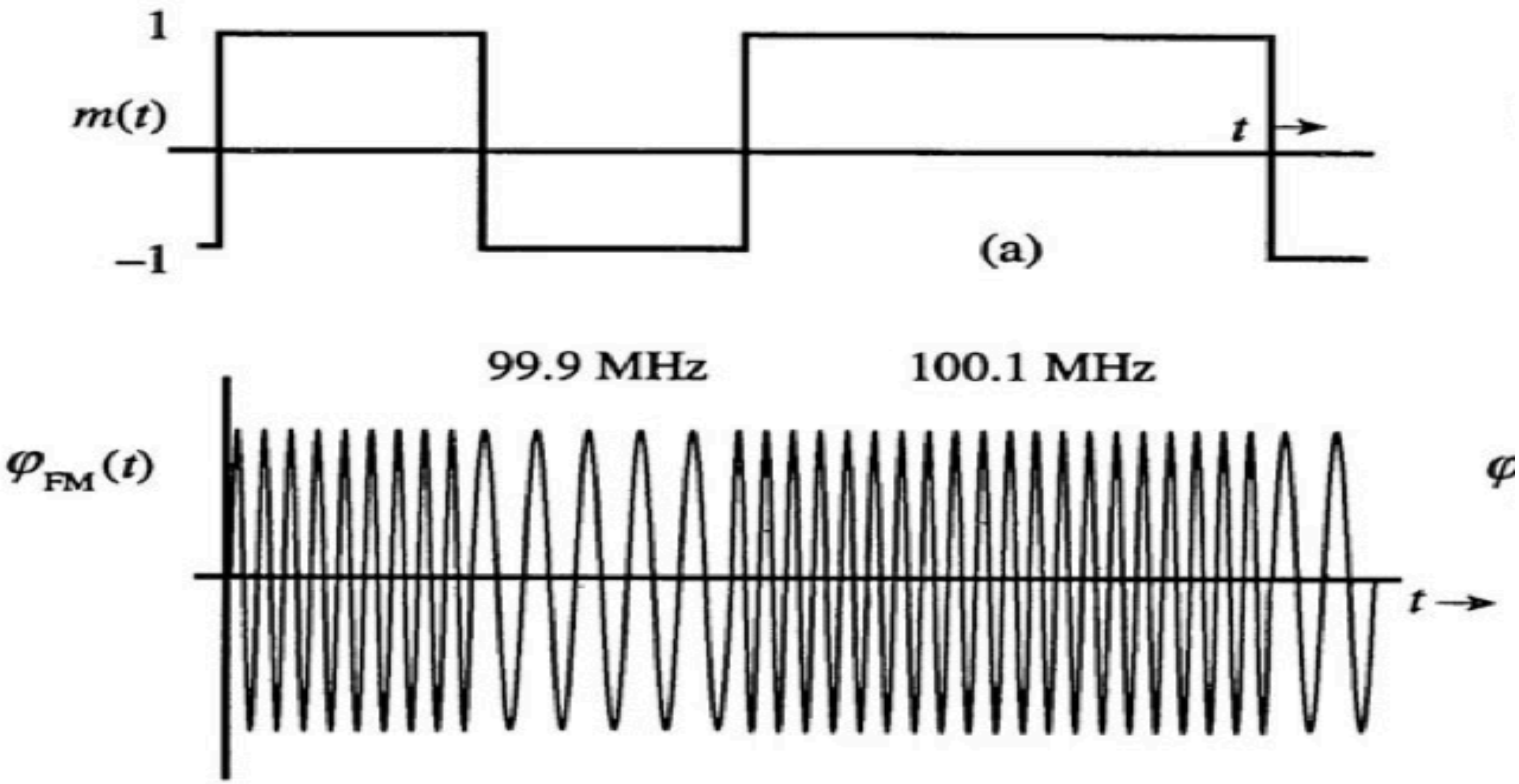
$$\theta_i(t) = 2\pi \int_{-\infty}^{t} f_i(\tau) d\tau \quad \text{and} \quad f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \theta_i(t)$$

Examples of FM Waveform



https://www.diffen.com/difference/Image:AM-FM-waves.gif

Examples of FM Waveform



FM in Time Domain

Frequency Modulation (FM)

Q: What is an FM signal?

Assume our carrier signal is:

$$v_c(t) = V_c \cos(2\pi f_c t)$$

$$f_c >> f_m$$

And our message (modulating) signal is:

$$v_m(t)$$

$$f_c > 10 f_m$$

A Frequency Modulated signal is when the message varies the frequency of the carrier.

Frequency Modulation (FM)

Q: What do we get?

The carrier frequency changes with the message signal. So the instantaneous frequency is:

Frequency Sensitivity (Hz/V)

Message signal

$$f_i(t) = f_c + k_f\,(v_m(t))$$

Instantaneous Frequency

Unmodulated Carrier Frequency

Note that the instantaneous frequency of an FM signal is directly related to the Message signal

Frequency Modulation (FM)

Q: What is the phase (angle) of a FM signal?

Integrating with respect to time and multiplying by 2 π to get the instantaneous phase:

$$\theta_i(t) = 2\pi \int_{-\infty}^{t} f_i(\tau) d\tau$$

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau) d\tau$$

Note that the instantaneous <u>phase</u> of an FM signal is directly related to the integral of the Message signal

Frequency Modulation (FM)

Q: What is a general expression for an FM signal?

We know that:

$$v_{FM}(t) = V_c \cos(\theta_i(t))$$

Substituting:

$$v_{FM}(t) = V_c \cos(2\pi f_c t + 2\pi k_f \int_{t-\infty}^{t} v_m(\tau)d\tau))$$

$$v_{FM}(t) = V_c \cos\left[2\pi(f_c t + k_f \int_{-\infty}^{t} v_m(\tau)d\tau)\right]$$

Frequency Modulation (FM)

Q: What is the expression for an FM signal modulated with a single tone? $v_m(t) = V_m \cos(2\pi f_m t)$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau) d\tau))$$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + \frac{k_f V_m}{f_m} \sin(2\pi f_m t))$$

Frequency deviation

$$\Delta f = k_f V_m = m_f f_m$$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t))$$

Modulation index

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$

Frequency Modulation (FM)

Q: What is the instantaneous frequency of an FM signal modulated with a single tone?

$$f_i(t) = f_c + k_f \ (v_m(t))$$

$$f_i(t) = f_c + k_f \ V_m \ \cos(2\pi \ f_m \ t)$$

Frequency deviation

$$f_i(t) = f_c + \Delta f \ \cos(2\pi \ f_m \ t)$$

$$\Delta f = k_f V_m = m_f f_m$$

Frequency Modulation (FM)

Q: What is the instantaneous phase of an FM signal modulated with a single tone?

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau)d\tau$$

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} V_m \cos(2\pi f_m \tau)d\tau$$

$$\theta_i(t) = 2\pi f_c t + \frac{k_f V_m}{f_m}\sin(2\pi f_m t)$$

Modulation index

$$\theta_i(t) = 2\pi f_c t + m_f \sin(2\pi f_m t)$$

$$m_f = \frac{k_f V_m}{f_m}$$

Frequency Modulation

Example: Draw the following FM signal?

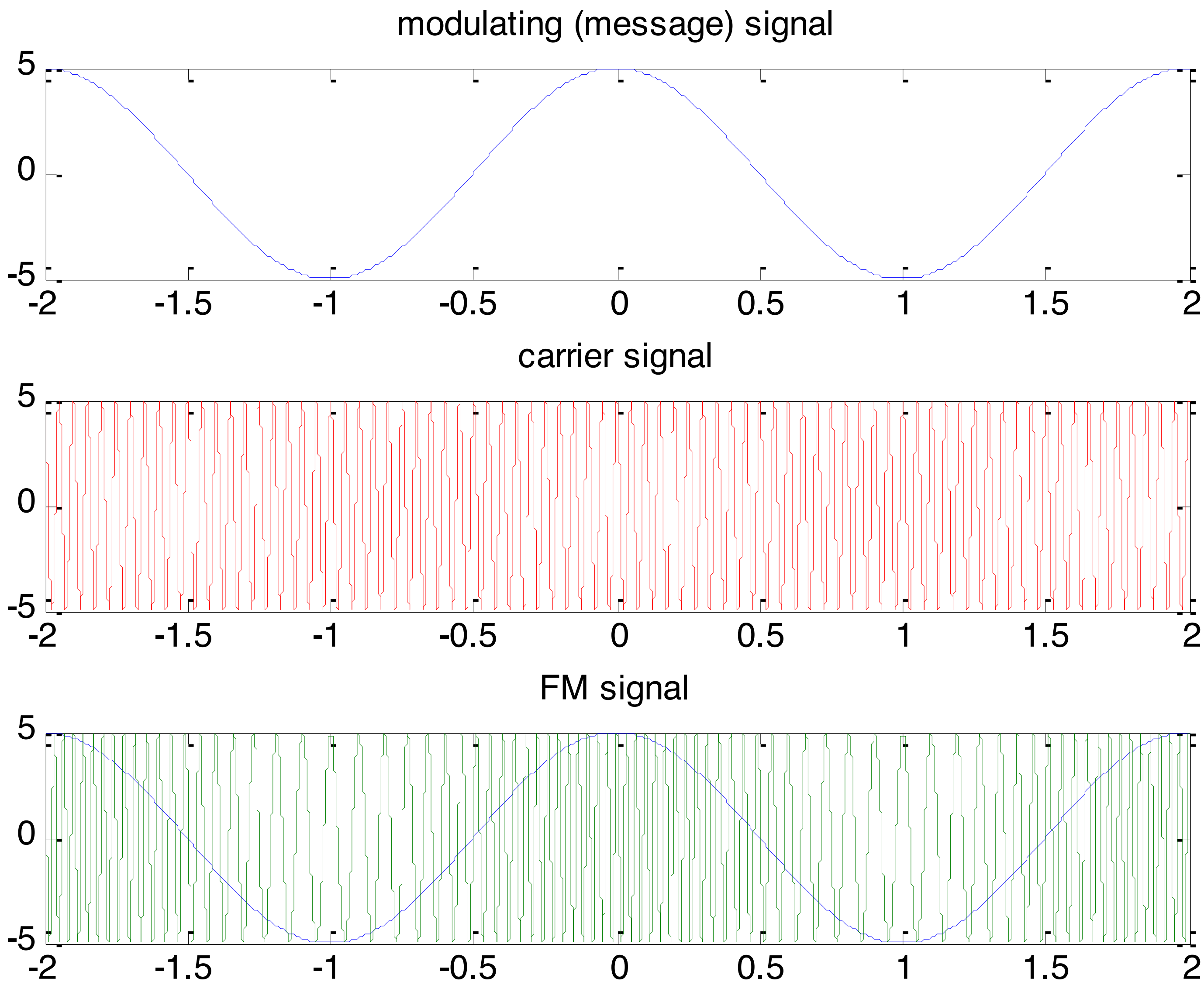
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = 5\cos(\pi\, t)$$

$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

Modulation index $k_f = 2 \Rightarrow \quad m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$


modulating (message) signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
carrier signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
FM signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Frequency Modulation

Example: Draw the following FM signal?

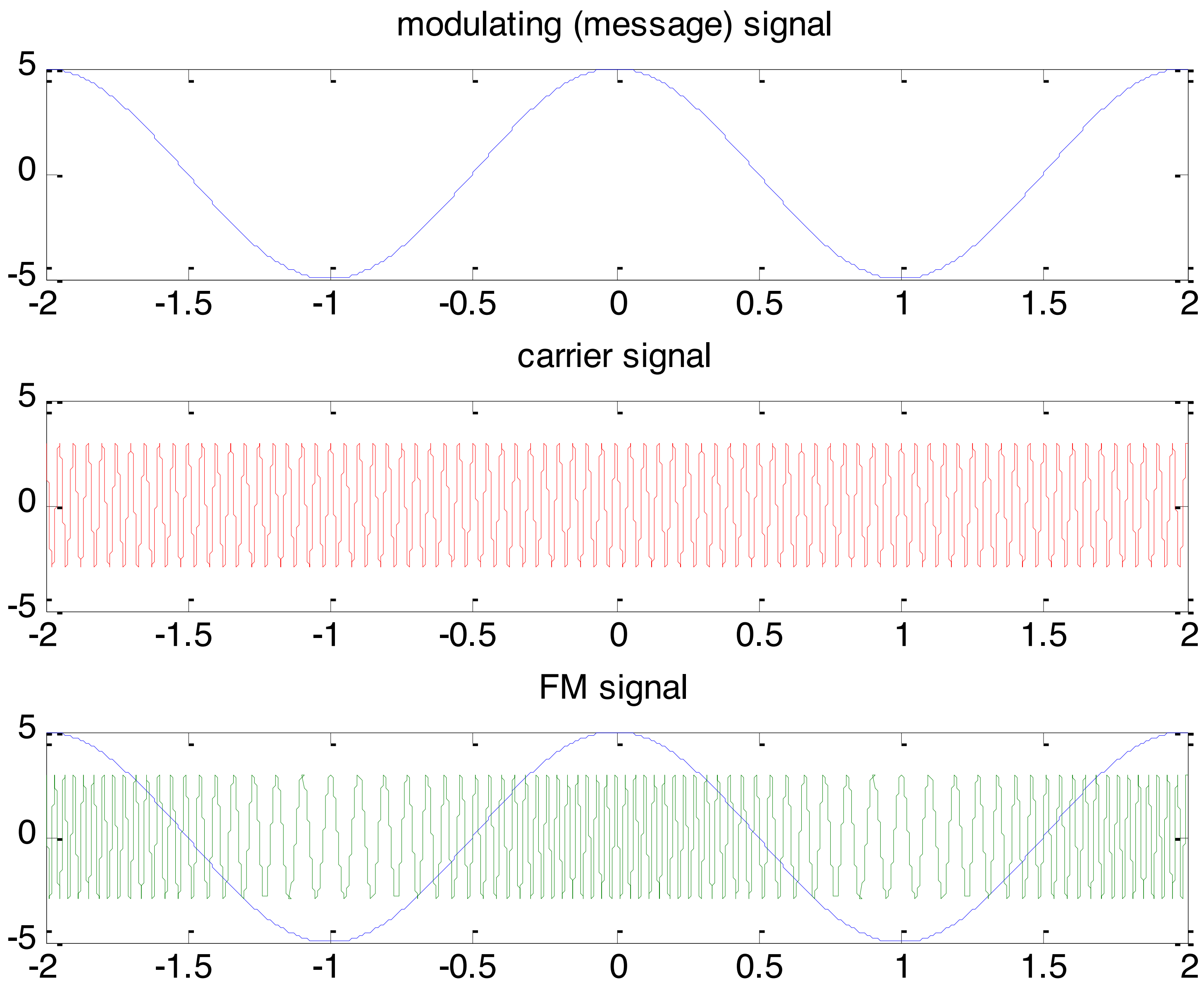
$$v_c(t) = 3\cos(40\pi\, t) \qquad v_m(t) = 5\cos(\pi\, t)$$

$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

Modulation index $k_f = 2 \Rightarrow \quad m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$


modulating (message) signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
carrier signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
FM signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Frequency Modulation

Example: Draw the following FM signal?

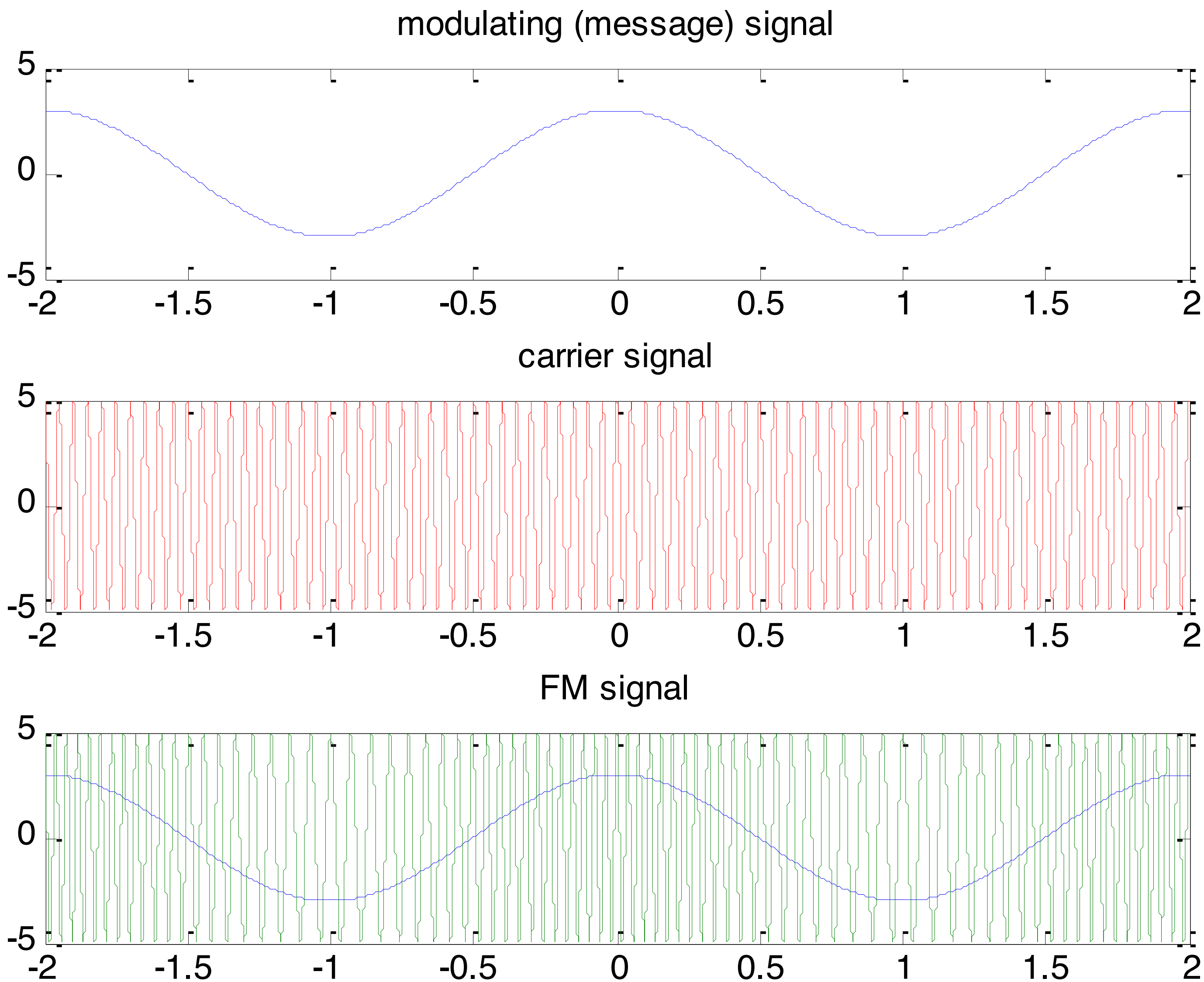
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = 3\cos(\pi\, t)$$

$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

Modulation index $k_f = 2 \Rightarrow \quad m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi\, f_m\, t))$$


modulating (message) signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
carrier signal
FM signal

Frequency Modulation

Example: Draw the following FM signal?

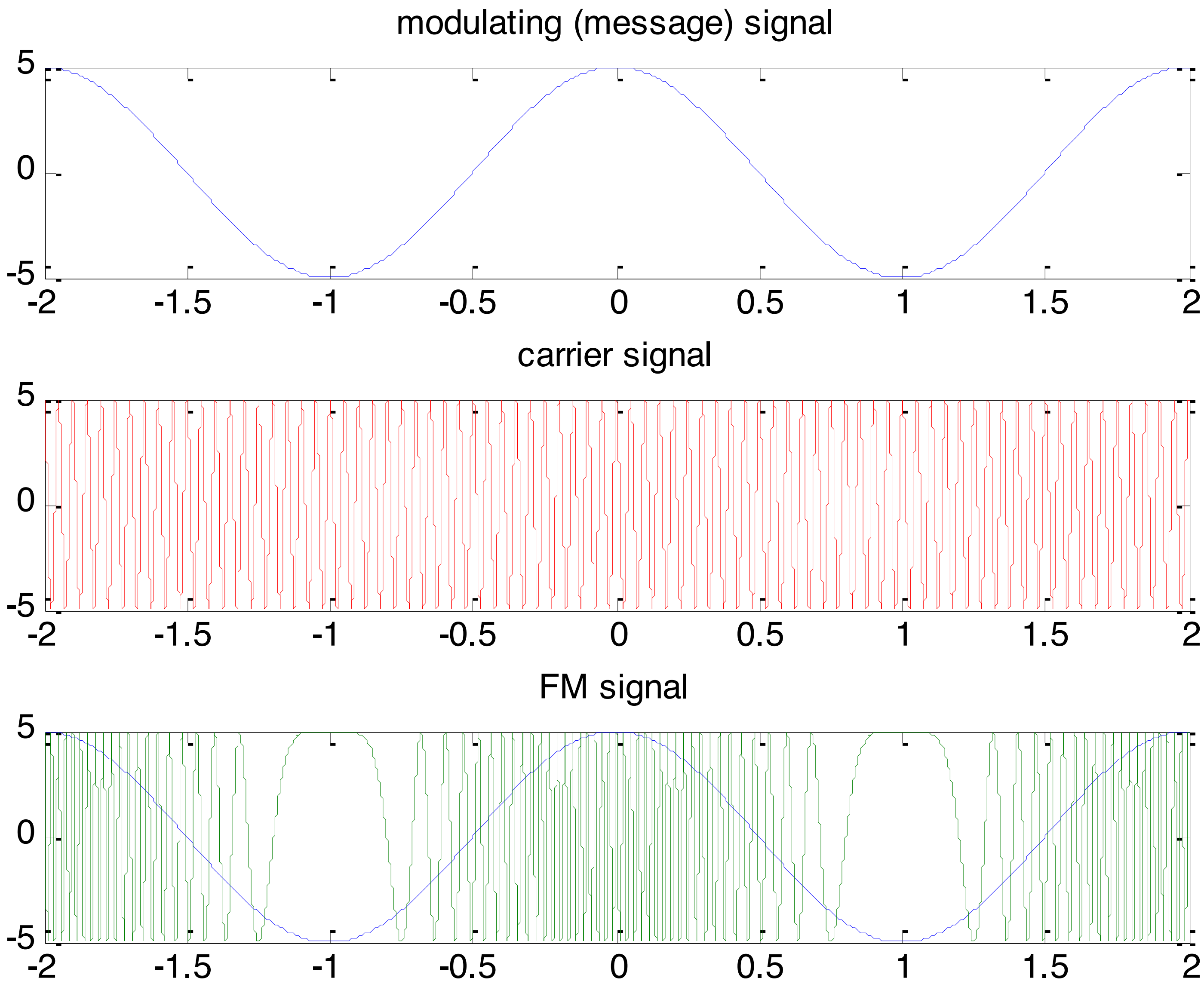
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = 5\cos(\pi\, t)$$

$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

Modulation index $k_f = 4 \Rightarrow \quad m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$


modulating (message) signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
carrier signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
FM signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Frequency Modulation

Example: What do you conclude?

Frequency Modulation: Summary

	Frequency Modulation (FM)
Instantaneous Phase	$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau)d\tau$
Instantaneous Frequency	$f_i(t) = f_c + k_f \ (v_m(t))$
Modulated signal	$v_{FM}(t) = V_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau)d\tau))$

Frequency Modulation: Summary

	Frequency Modulation (FM)
Sensitivity	$k_f (Hz/V)$
Frequency deviation	$\Delta f = k_f V_m = m_f f_m (Hz)$
Modulation index	$m_f = \frac{k_f V_m}{f_m}$

Angle Modulation: Summary for Single tone modulation

	Frequency Modulation (FM)
Instantaneous Phase	$\theta_i(t) = 2\pi f_c t + m_f \sin(2\pi f_m t)$
Instantaneous Frequency	$f_i(t) = f_c + \Delta f \cos(2\pi f_m t)$
Modulated signal	$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$

$$v_m(t) = V_m \cos(2\pi f_m t)$$

Frequency Modulation: Frequency Domain

FM Spectrum

Q: How does the spectrum of an FM signal look?

Going back to the defining equation:

$$v_{FM}(t) = V_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} v_m(\tau) d\tau))$$

Assuming single tone modulation: $v_m(t) = V_m \cos(2\pi f_m t)$

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$

$$v_{FM}(t) = V_c \cos(2\pi f_c t) \cos\left(m_f \sin(2\pi f_m t)\right) - V_c \sin(2\pi f_c t) \sin\left(m_f \sin(2\pi f_m t)\right)$$

Narrow band FM Spectrum

Q: What if the modulation index m_f is small ?

Then:

$$\cos\left(m_f \ \sin(2\pi \ f_m \ t)\right) \approx 1$$

$$\sin\left(m_f \ \sin(2\pi \ f_m \ t)\right) \approx m_f \ \sin(2\pi \ f_m \ t)$$

Substituting:

$$v_{FM}(t) \approx \ V_c \cos(2\pi f_c t) - V_c \sin(2\pi f_c t) m_f \sin(2\pi \ f_m \ t)$$

$$v_{FM}(t) \approx \ V_c \cos(2\pi f_c t) - m_f V_c \sin(2\pi f_c t) \sin(2\pi \ f_m \ t)$$

$$v_{FM}(t) \approx \ V_c \cos(2\pi f_c t) + \frac{m_f V_c}{2}\left(\cos(2\pi(f_c + f_m)t) - \cos(2\pi(f_c - f_m)t)\right)$$

Compare NB- FM to AM Spectrum

Q: What are the similarities and differences?

Covert to the frequency domain

$$v_{AM}(t) = V_c \cos(2\pi\ f_c\ t) + \frac{mV_c}{2}\left(\cos(2\pi\ (f_c + f_m)\ t) + \cos(2\pi\ (f_c - f_m)\ t)\right)$$

$$v_{FM}(t) \approx\ V_c \cos(2\pi f_c t) + \frac{m_f V_c}{2}\left(\cos(2\pi(f_c + f_m)t) - \cos(2\pi(f_c - f_m)t)\right)$$

Narrow band FM Spectrum

Example: Sketch the spectrum of the following FM signal?

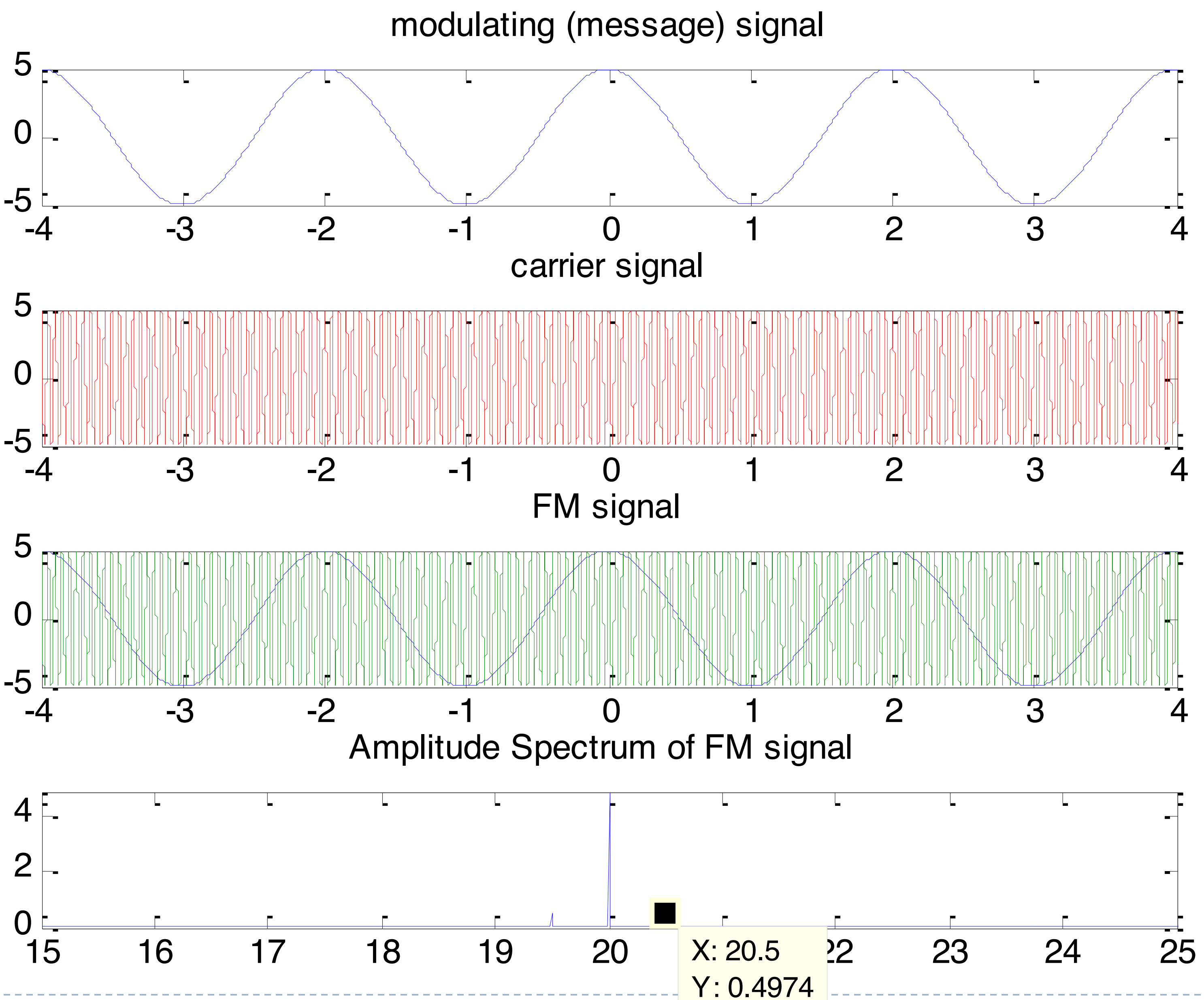
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = 5\cos(\pi\, t)$$

$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

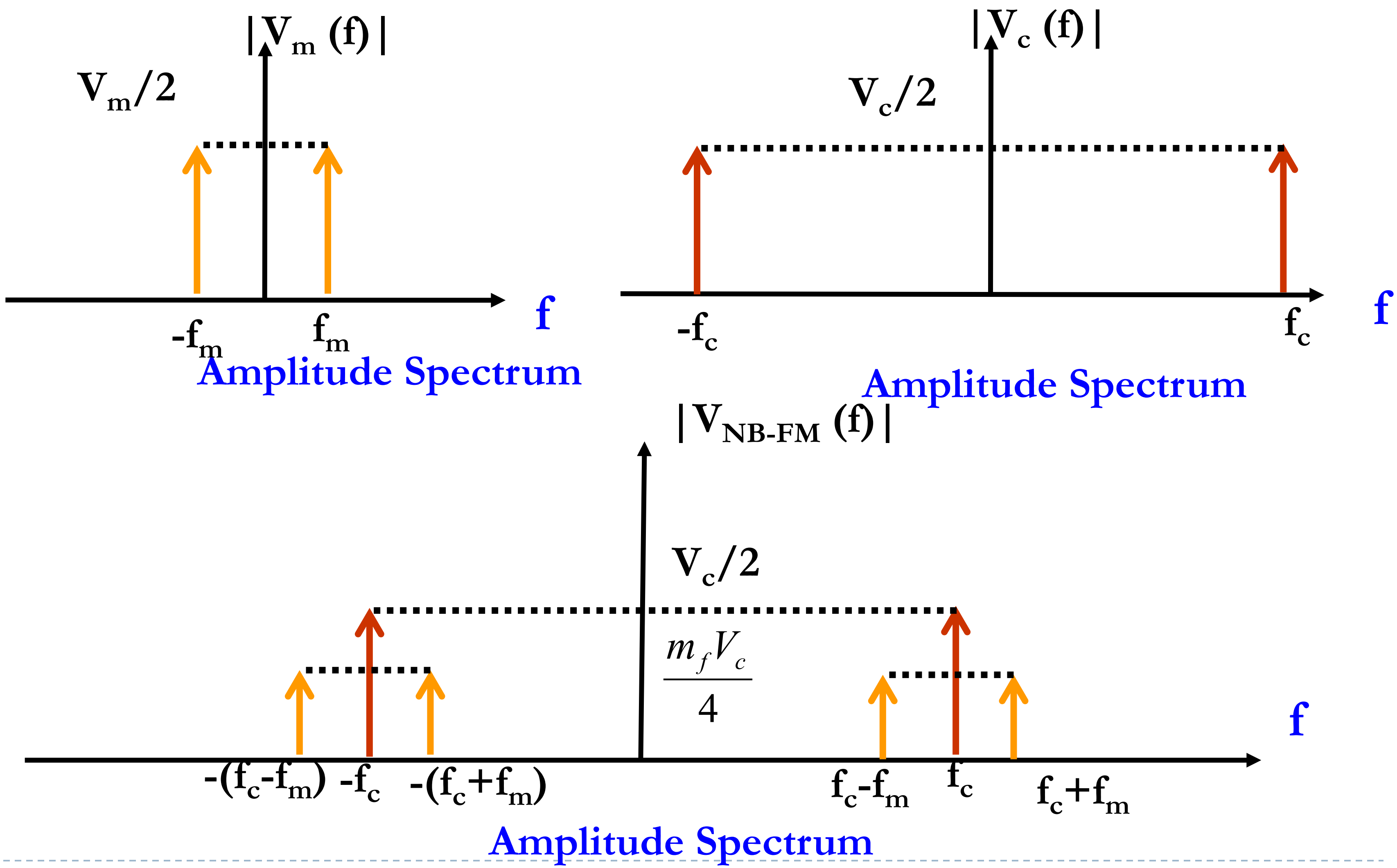
$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

Modulation index $k_f = 0.02 \Rightarrow m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

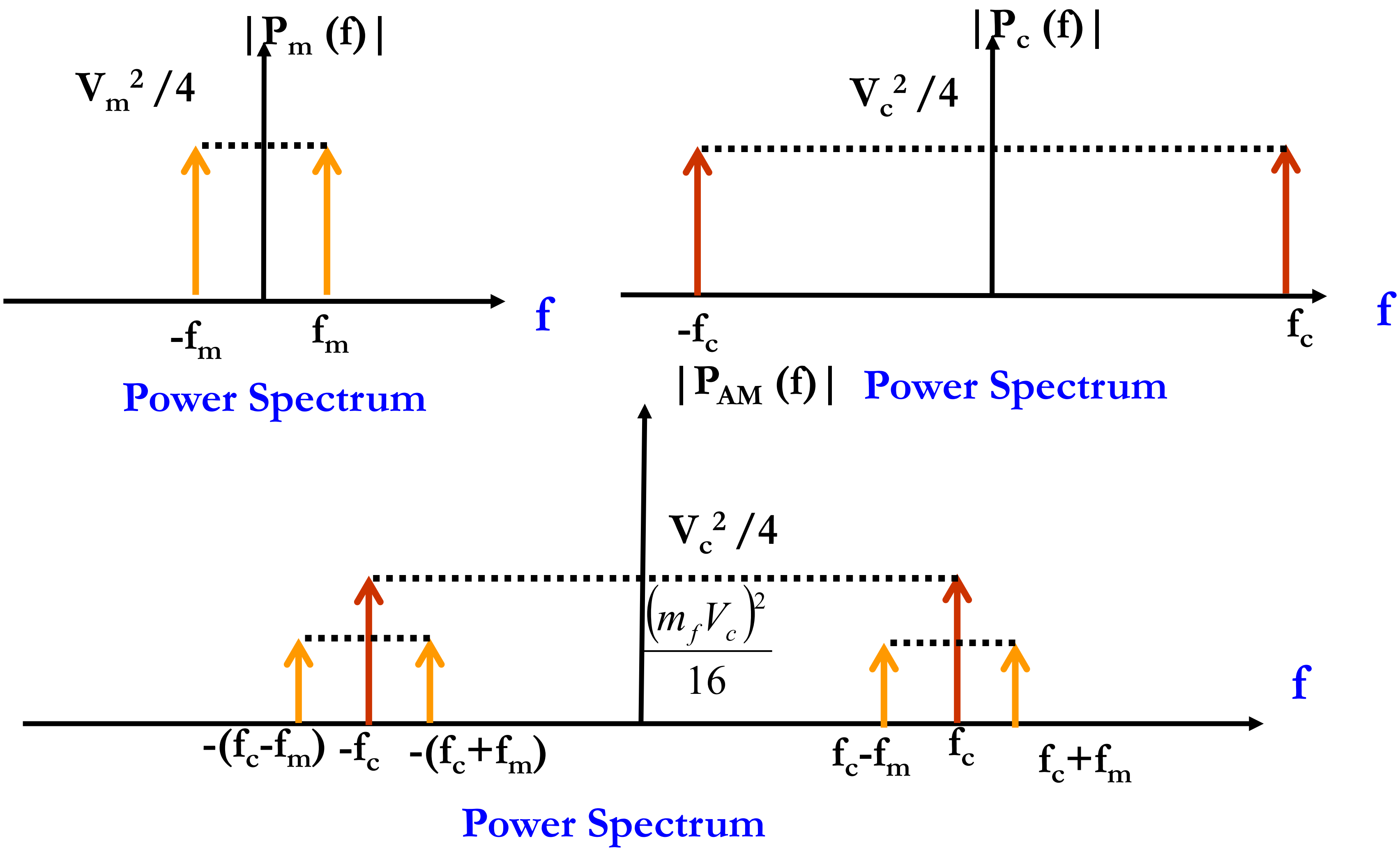
$$v_{FM}(t) \approx V_c \cos(2\pi f_c t) + \frac{m_f V_c}{2}\left(\cos(2\pi(f_c + f_m)t) - \cos(2\pi(f_c - f_m)t)\right)$$


modulating (message) signal
carrier signal
FM signal
Amplitude Spectrum of FM signal
X: 20.5
Y: 0.4974

Narrow band FM Spectrum



Narrow band FM Spectrum



NB-FM Bandwidth

Q: What is the Bandwidth of an NB-FM signal?

The Bandwidth of the NB-FM signal is defined as:

$$BW = f_{USB} - f_{LSB}$$

$$BW = 2f_m$$

Wide band FM Spectrum

Q: What if the modulation index m_f is large ?

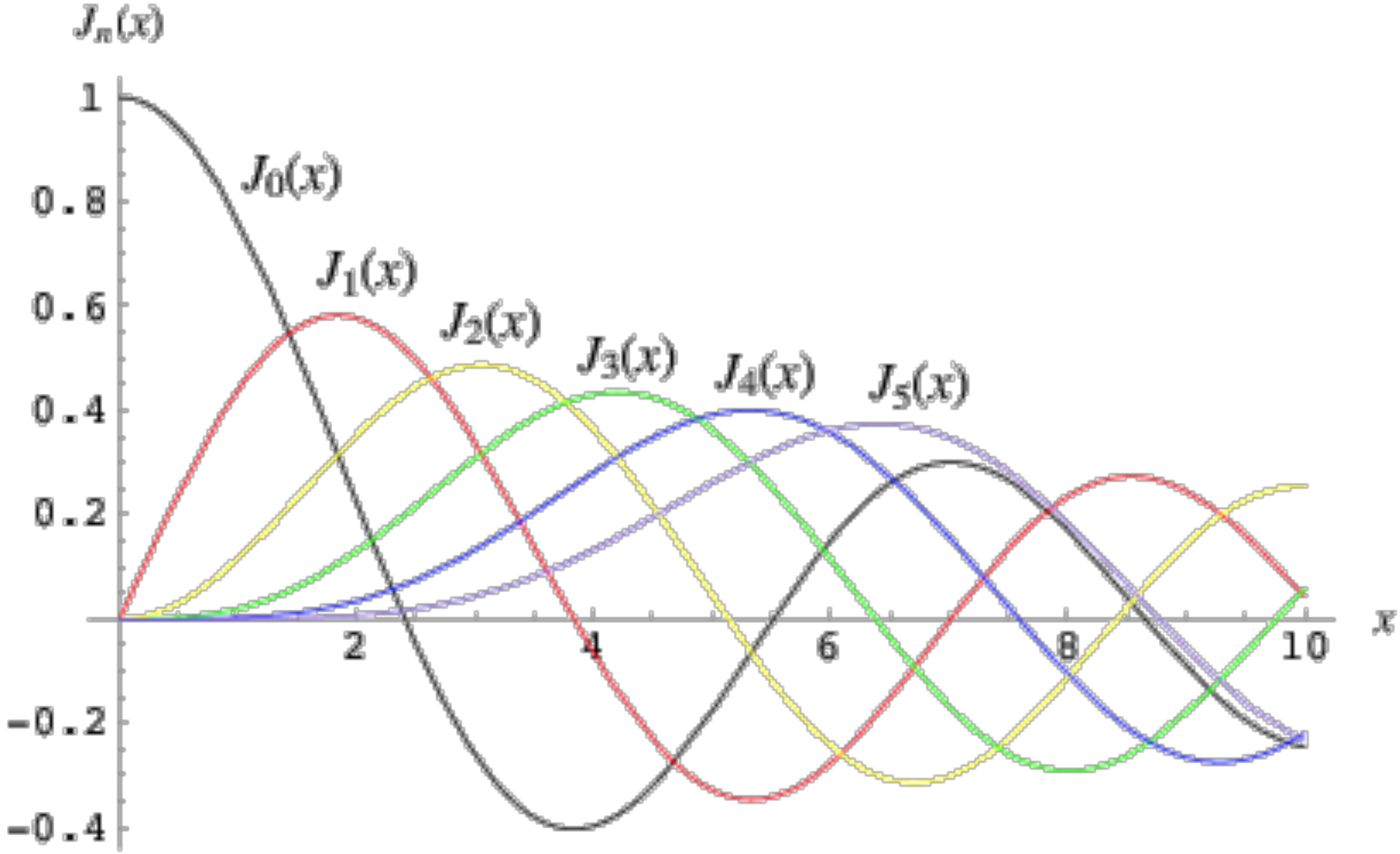
We cannot use the approximation:

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t))$$

Using Bessel functions:

$$v_{FM}(t) = V_c \cos(2\pi f_c t + m_f \sin(2\pi f_m t)) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f) \cos\left[2\pi(f_c + nf_m)t\right]$$

where



$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$
2
4
6
8
10
x

Wide band FM Spectrum

Q: Expand the expression for WB-FM?

$$v_{FM}(t) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f)\cos\left[2\pi(f_c + nf_m)t\right]$$

$$v_{FM}(t) = V_c \cdot \begin{Bmatrix} \ldots\ldots \\ + J_{-3}(m_f)\cos\left[2\pi(f_c - 3f_m)t\right] \\ + J_{-2}(m_f)\cos\left[2\pi(f_c - 2f_m)t\right] \\ + J_{-1}(m_f)\cos\left[2\pi(f_c - f_m)t\right] \\ + J_0(m_f)\cos\left[2\pi(f_c)t\right] \\ + J_1(m_f)\cos\left[2\pi(f_c + f_m)t\right] \\ + J_2(m_f)\cos\left[2\pi(f_c + 2f_m)t\right] \\ + J_3(m_f)\cos\left[2\pi(f_c + 3f_m)t\right] \\ + \ldots\ldots\ldots\ldots \end{Bmatrix}$$

$$J_n(m_f) = (-1)^n J_{-n}(m_f)$$

Wide band FM Spectrum

Example: Sketch the spectrum of the following FM signal?

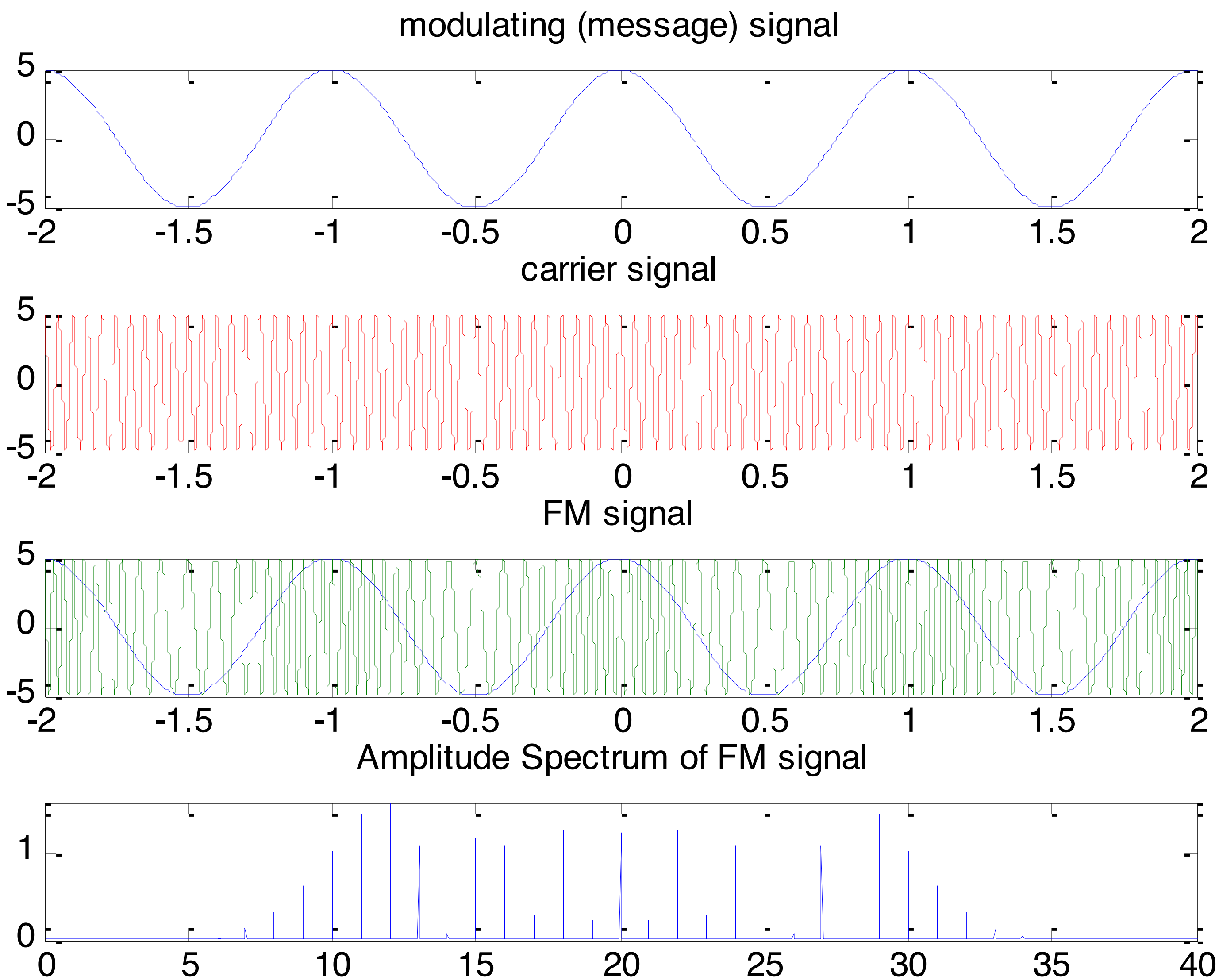
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = 5\cos(2\pi\, t)$$

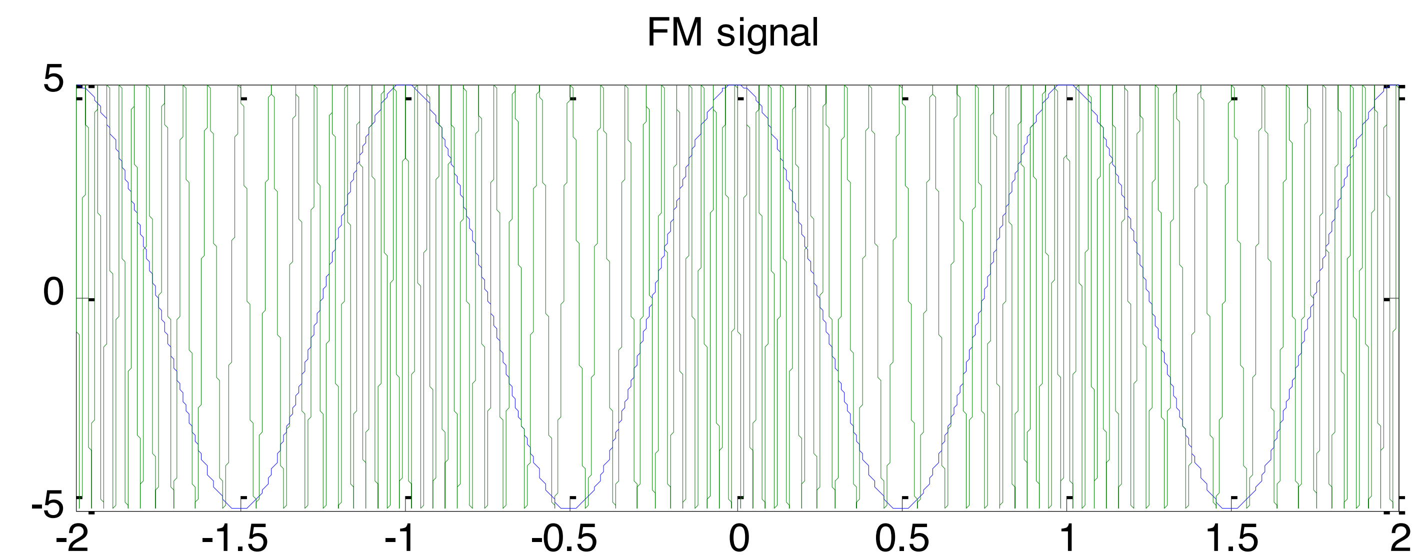
$$f_c = \ldots\ldots \qquad f_m = \ldots\ldots$$

$$V_c = \ldots\ldots \qquad V_m = \ldots\ldots$$

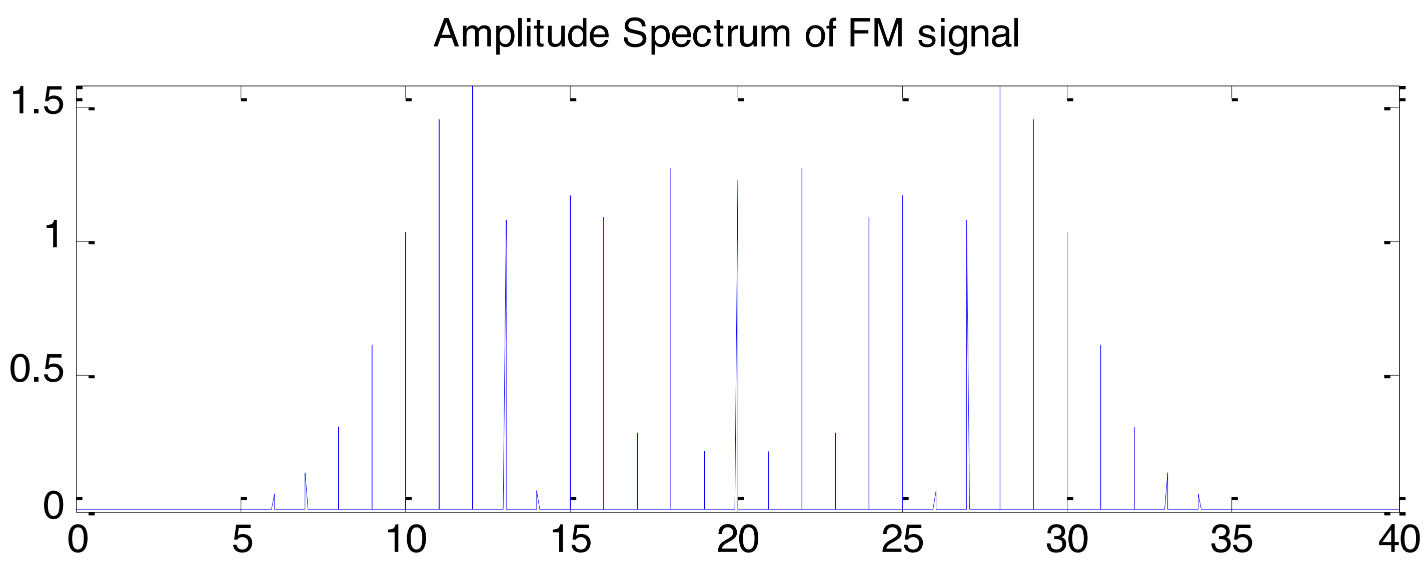
Modulation index $k_f = 2 \Rightarrow \quad m_f = \dfrac{k_f V_m}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f)\cos\left[2\pi(f_c + nf_m)t\right]$$


modulating (message) signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
carrier signal
FM signal
Amplitude Spectrum of FM signal
1
0
5
10
15
20
25
30
35
40



FM signal
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



Amplitude Spectrum of FM signal
1.5
1
0.5
0
0
5
10
15
20
25
30
35
40

Examples of FM (tone modulation)

Example: Sketch the spectrum for the following FM signal?

$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = \cos(2\pi\, t)$$

$$k_f = 2$$

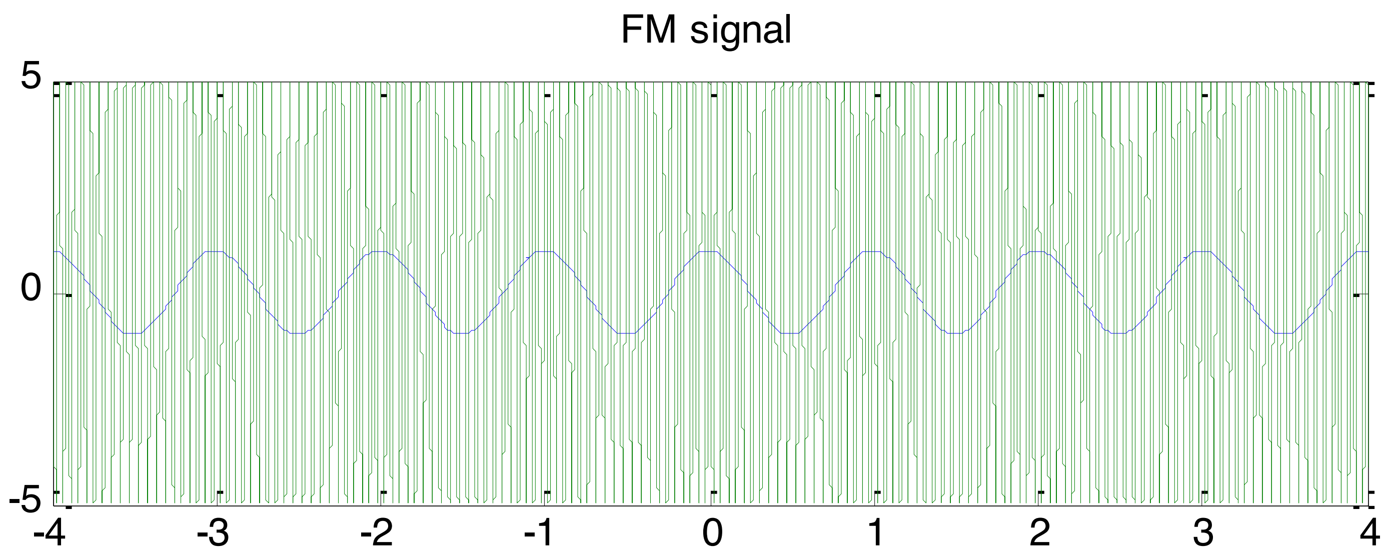
Frequency Deviation $\Delta f = k_f V_m = \ldots\ldots$

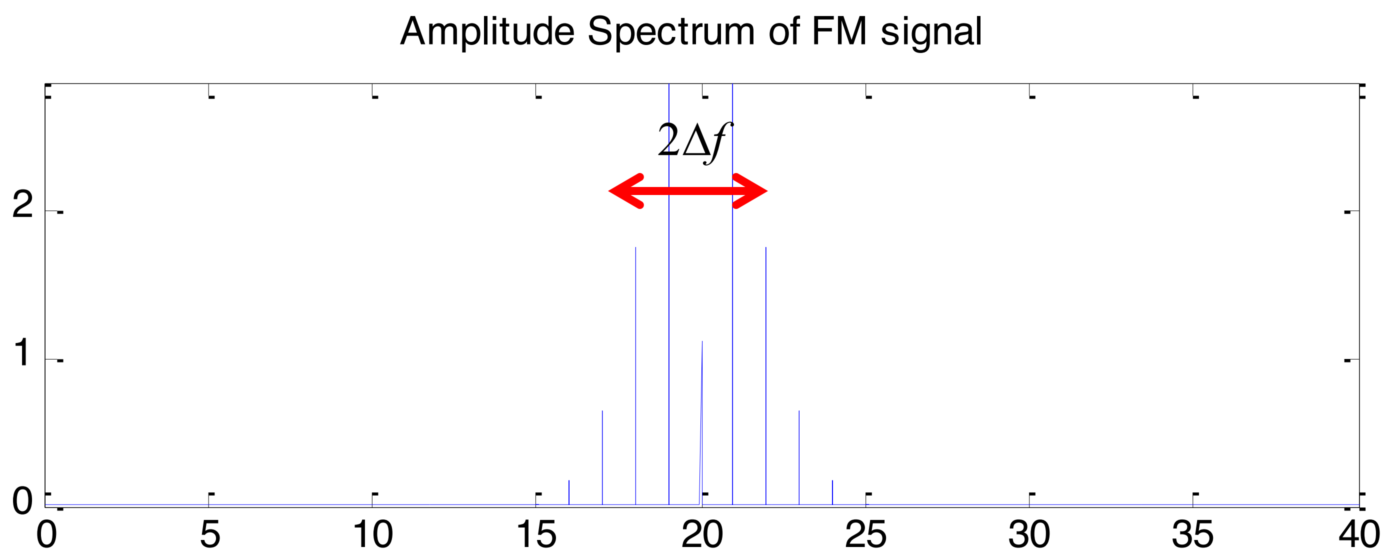
Modulation index $m_f = \dfrac{\Delta f}{f_m} = \ldots\ldots$

$$v_{FM}(t) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f)\cos\left[2\pi(f_c + nf_m)t\right]$$

$$v_{FM}(t) = 5\sum_{n=-\infty}^{\infty} J_n(2)\cos\left[2\pi(20+n)t\right]$$

$$v_{FM}(t) = 5.\left\{\begin{array}{l} J_{-5}(2)\cos[2\pi(15)t] \\ +J_{-4}(2)\cos[2\pi(16)t] \\ +J_{-3}(2)\cos[2\pi(17)t] \\ +J_{-2}(2)\cos[2\pi(18)t] \\ +J_{-1}(2)\cos[2\pi(19)t] \\ +J_{0}(2)\cos[2\pi(20)t] \\ +J_{1}(2)\cos[2\pi(21)t] \\ +J_{2}(2)\cos[2\pi(22)t] \\ +J_{3}(2)\cos[2\pi(23)t] \\ +J_{4}(2)\cos[2\pi(24)t] \\ +J_{5}(2)\cos[2\pi(25)t] \end{array}\right\} = \left\{\begin{array}{l} -0.0352\cos[2\pi(15)t] \\ +0.1700\cos[2\pi(16)t] \\ -0.6447\cos[2\pi(17)t] \\ +1.7642\cos[2\pi(18)t] \\ -2.8836\cos[2\pi(19)t] \\ +1.1195\cos[2\pi(20)t] \\ +2.8836\cos[2\pi(21)t] \\ +1.7642)\cos[2\pi(22)t] \\ +0.6447\cos[2\pi(23)t] \\ +0.1700\cos[2\pi(24)t] \\ +0.0352\cos[2\pi(25)t] \end{array}\right\}$$


FM signal
5
0
-5
-4
-3
-2
-1
0
1
2
3
4


Amplitude Spectrum of FM signal
$2\Delta f$
2
1
0
0
5
10
15
20
25
30
35
40

```
fm=0.5;  fc=20;  Vm=5;  Vc=5;  kf=0.2;

mf=kf*Vm/fm

ts=0.0001

t=-40:ts:40;

vfm =Vc * cos( (2*pi*fc*t) + mf*sin(2*pi*fm*t));

N=length(t);

f = [-N/2:N/2-1]/(N*ts);

z = fftshift(fft(vfm))/N;

plot(f, abs(z))

xlim([fc-fm*10 fc+fm*10])
```

Examples of FM (tone modulation)

Example: Sketch the spectrum for the following FM signals?

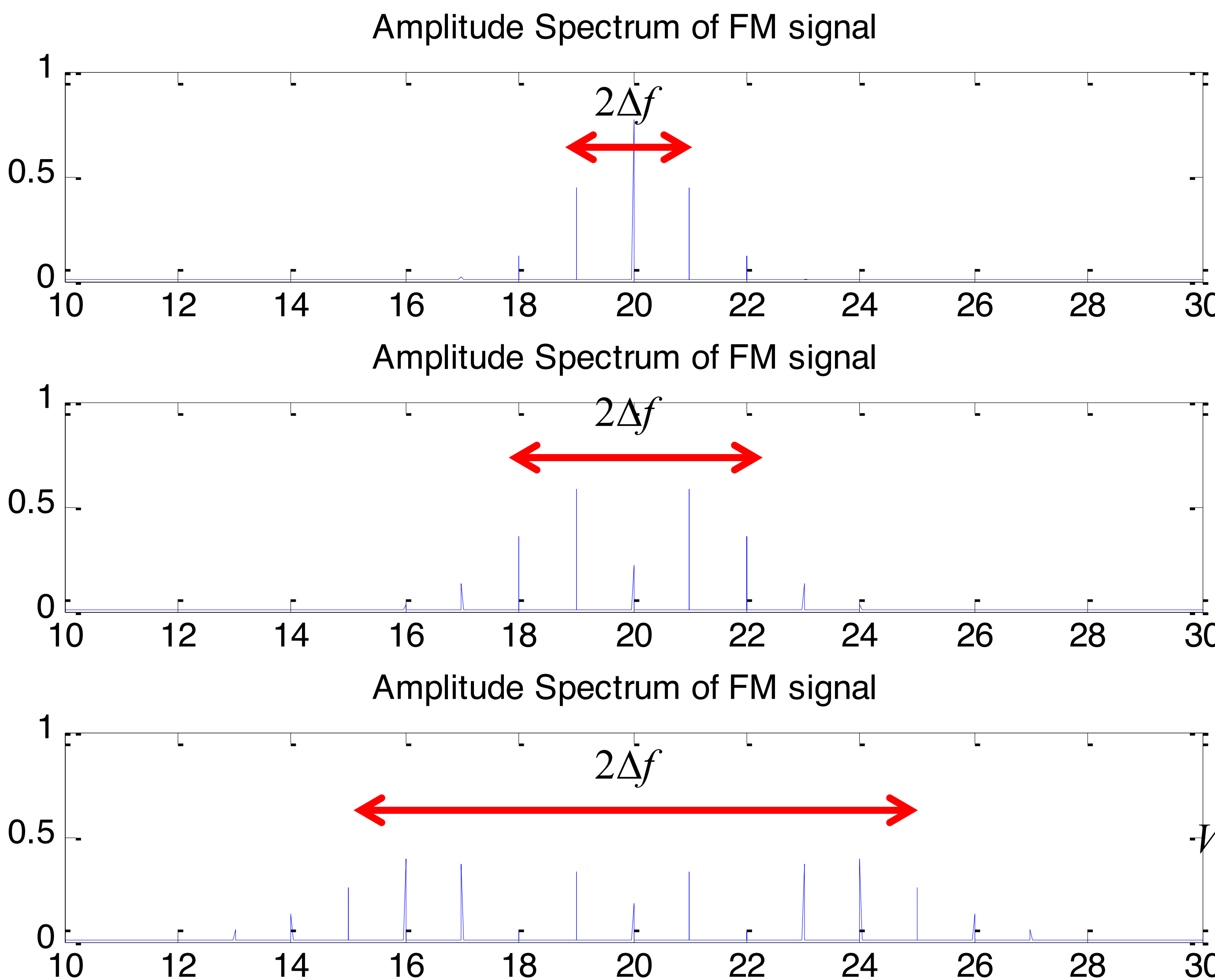
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = V_m \cos(2\pi\, t)$$

$$V_m = 1,2,5 \qquad k_f = 1$$

Frequency Deviation $\Delta f = k_f V_m = \ldots\ldots,\ldots\ldots,\ldots\ldots$

Modulation index $m_f = \dfrac{\Delta f}{f_m} = \ldots\ldots,\ldots\ldots,\ldots\ldots$

$$v_{FM}(t) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f)\cos\left[2\pi(f_c + nf_m)t\right]$$



$V_m = 1, \quad \Delta f = 1, \quad m_f = 1$

$V_m = 2, \quad \Delta f = 2, \quad m_f = 2$

$V_m = 5, \quad \Delta f = 5, \quad m_f = 5$

Examples of FM (tone modulation)

Example: Sketch the spectrum for the following FM signals?

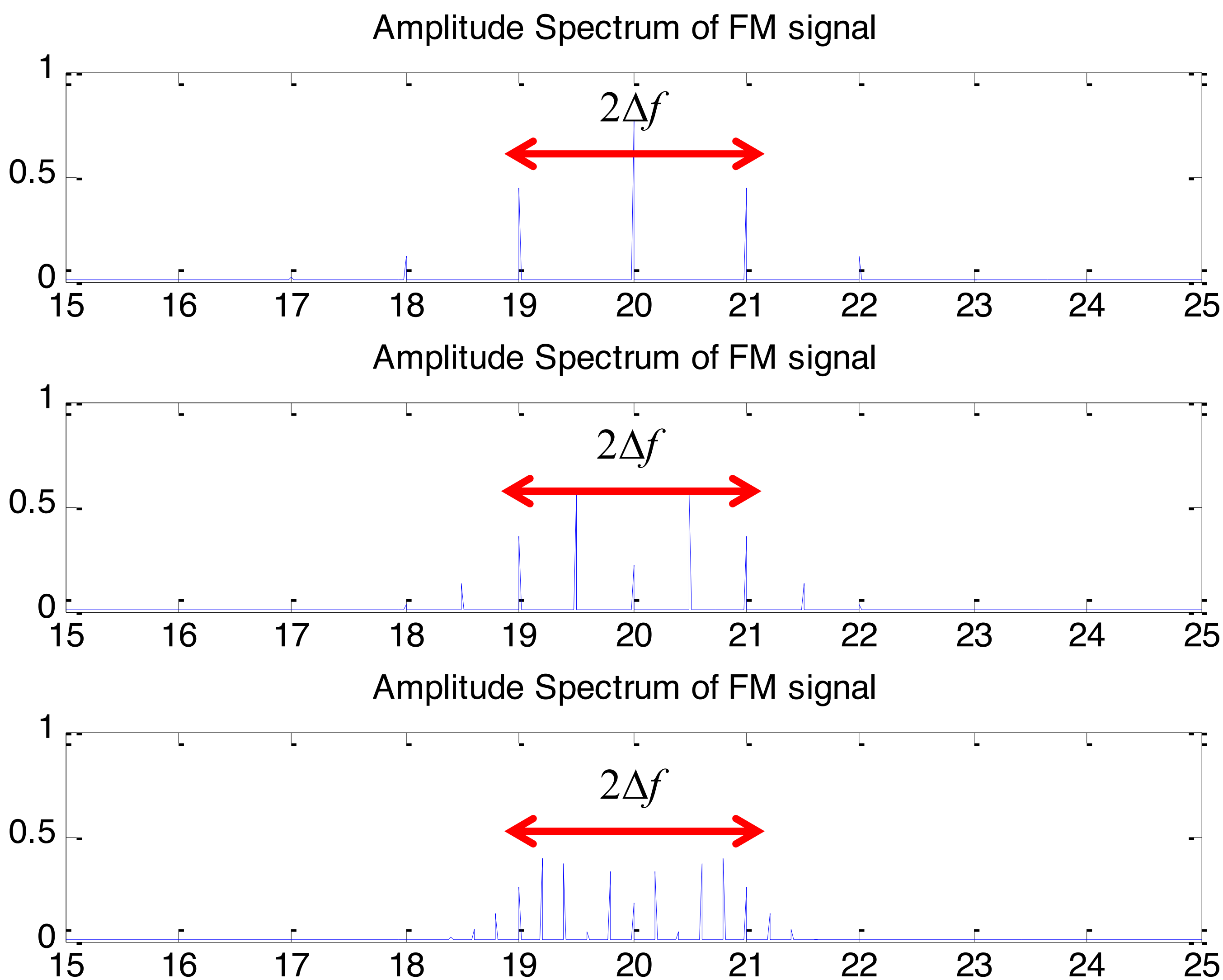
$$v_c(t) = 5\cos(40\pi\, t) \qquad v_m(t) = \cos(2\pi f_m t)$$

$$f_m = 1, \quad 0.5, \quad 0.2 \qquad k_f = 1$$

Frequency Deviation $\Delta f = k_f V_m = \ldots\ldots,\ldots\ldots,\ldots\ldots$

Modulation index $m_f = \dfrac{\Delta f}{f_m} = \ldots\ldots,\ldots\ldots,\ldots\ldots$

$$v_{FM}(t) = V_c \sum_{n=-\infty}^{\infty} J_n(m_f)\cos\left[2\pi(f_c + nf_m)t\right]$$



$$f_m = 1, \quad \Delta f = 1, \quad m_f = 1$$

$$f_m = 0.5, \quad \Delta f = 1, \quad m_f = 2$$

$$f_m = 0.2, \quad \Delta f = 1, \quad m_f = 5$$

Examples of FM (tone modulation)

Q: Compare the results on slides 62 and 64 for the same modulation index. What do you conclude?

FM Bandwidth

Q: What is the Bandwidth of an FM signal?

From Bessel Coefficient Table:

For a given modulation index m_f we determine the significant sidebands from the table.

If we have n significant sideband then:

$$BW = 2n\, f_m$$

FM Bandwidth

Q: What is the approxiamte Bandwidth of an FM signal?

Carlson's Rule:

states that nearly all (~98%) of the power of an FM lies within a bandwidth

$$BW \approx 2(\Delta f + f_m)$$

$$BW \approx 2 f_m (m_f + 1)$$

But we have shown earlier that for narrow-band, the FM bandwidth is $2B$ Hz. This indicates that a better bandwidth estimate is

$$B_{FM} = 2(\Delta f + B) \tag{5.13a}$$

$$= 2\left(\frac{k_f m_p}{2\pi} + B\right) \tag{5.13b}$$

This is precisely the result obtained by Carson,[1] who investigated this problem rigorously for tone modulation [sinusoidal $m(t)$]. This formula goes under the name **Carson's rule** in the literature. Observe that for a truly wide-band case, where $\Delta f \gg B$, Eqs. (5.13) can be approximated as

$$B_{FM} \approx 2\Delta f \qquad \Delta f \gg B \tag{5.14}$$

Because $\Delta\omega = k_f m_p$, this formula is precisely what the pioneers had used for FM bandwidth. The only mistake was in thinking that this formula will hold for all cases, especially for the narrow-band case, where $\Delta f \ll B$.

We define a deviation ratio β as

$$\beta = \frac{\Delta f}{B} \tag{5.15}$$

Carson's rule can be expressed in terms of the deviation ratio as

$$B_{FM} = 2B(\beta + 1) \tag{5.16}$$

The deviation ratio controls the amount of modulation and, consequently, plays a role similar to the modulation index in AM. Indeed, for the special case of tone-modulated FM, the deviation ratio β is called the **modulation index.**

FM Total transmitted Power

Q: What is the total power in an FM signal?

Since the amplitude of an FM signal is constant and equal to V_c then the total power in an FM wave is equal to:

$$P_T = \frac{V_c^{\ 2}}{2}$$

Remember that:

$$P_T = P_C + P_{SB}$$

FM Total transmitted Power

Q: What is the power in the carrier and side bands?

For single tone modulation, the power is the carrier is:

$$P_C = \frac{\left(J_0(m_f).V_c\right)^2}{2} = \left(J_0(m_f)\right)^2.\frac{V_c^{\;2}}{2} = \left(J_0(m_f)\right)^2.P_T$$

The power in the nth side bands is:

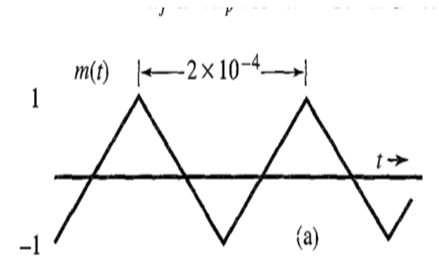
$$P_{SBn} = \frac{\left(J_n(m_f).V_c\right)^2}{2} = \left(J_n(m_f)\right)^2.\frac{V_c^{\;2}}{2} = \left(J_n(m_f)\right)^2.P_T$$

The total power in the sidebands is:

$$P_{SB} = \sum_{n=-\infty}^{\infty}\left(J_n(m_f)\right)^2 P_T$$

EXAMPLE 5.3 (a) Estimate B_{FM} and B_{PM} for the modulating signal $m(t)$ in Fig. 5.4a for $k_f = 2\pi \times 10^5$ and $k_p = 5\pi$.

(b) Repeat the problem if the amplitude of $m(t)$ is doubled [if $m(t)$ is multiplied by 2].



Repeat Example 5.3 if $m(t)$ is time-expanded by a factor of 2; that is, if the period of $m(t)$ is 4×10^{-4}.

(a) The peak amplitude of $m(t)$ is unity. Hence, $m_p = 1$. We now determine the essential bandwidth B of $m(t)$. It is left as an exercise for the reader to show that the Fourier series for this periodic signal is given by

$$m(t) = \sum_n C_n \cos n\omega_0 t \qquad \omega_0 = \frac{2\pi}{2 \times 10^{-4}} = 10^4 \pi$$

where

$$C_n = \begin{cases} \dfrac{8}{\pi^2 n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

It can be seen that the harmonic amplitudes decrease rapidly with n. The third harmonic is only 11% of the fundamental, and the fifth harmonic is only 4% of the fundamental. This means the third and fifth harmonic powers are 1.21 and 0.16%, respectively, of the fundamental component power. Hence, we are justified in assuming the essential bandwidth of $m(t)$ as the frequency of the third harmonic, that is, $3(10^4/2)$ Hz. Thus,

$$B = 15 \text{ kHz}$$

For FM:

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}(2\pi \times 10^5)(1) = 100 \text{ kHz}$$

and

$$B_{FM} = 2(\Delta f + B) = 230 \text{ kHz}$$

Alternately, the deviation ratio β is given by

$$\beta = \frac{\Delta f}{B} = \frac{100}{15}$$

and

$$B_{FM} = 2B(\beta + 1) = 30\left(\frac{100}{15} + 1\right) = 230 \text{ kHz}$$

(b) Doubling $m(t)$ doubles its peak value. Hence, $m_p = 2$. But its bandwidth is unchanged so that $B = 15$ kHz.

For FM:

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}(2\pi \times 10^5)(2) = 200 \text{ kHz}$$

and

$$B_{FM} = 2(\Delta f + B) = 430 \text{ kHz}$$

Alternately, the deviation ratio β is given by

$$\beta = \frac{\Delta f}{B} = \frac{200}{15}$$

and

$$B_{FM} = 2B(\beta + 1) = 30\left(\frac{200}{15} + 1\right) = 430 \text{ kHz}$$

Recall that time expansion of a signal by a factor of 2 reduces the signal spectral width (bandwidth) by a factor of 2. We can verify this by observing that the fundamental frequency is now 2.5 kHz, and its third harmonic is 7.5 kHz. Hence, $B = 7.5$ kHz, which is half the previous bandwidth. Moreover, time expansion does not affect the peak amplitude so that $m_p = 1$. However, m'_p is halved, that is, $m'_p = 10,000$.

For FM:

$$\Delta f = \frac{1}{2\pi} k_f m_p = 100 \text{ kHz}$$

$$B_{FM} = 2(\Delta f + B) = 2(100 + 7.5) = 215 \text{ kHz}$$

An angle-modulated signal with carrier frequency $\omega_c = 2\pi \times 10^5$ is described by the equation

$$\varphi_{EM}(t) = 10\cos(\omega_c t + 5\sin 3000t + 10\sin 2000\pi t)$$

(a) Find the power of the modulated signal.
(b) Find the frequency deviation Δf.
(c) Find the deviation ratio β.

(e) Estimate the bandwidth of $\varphi_{EM}(t)$.

The signal bandwidth is the highest frequency in $m(t)$ (or its derivative). In this case $B = 2000\pi/2\pi = 1000$ Hz.

(a) The carrier amplitude is 10, and thc power is

$$P = 10^2/2 = 50$$

(b) To find the frequency deviation Δf, we find the instantaneous frequency ω_i, given by

$$\omega_i = \frac{d}{dt}\theta(t) = \omega_c + 15,000\cos 3000t + 20,000\pi\cos 2000\pi t$$

The carrier deviation is $15,000\cos 3000t + 20,000\pi\cos 2000\pi t$. The two sinusoids will add in phase at some point, and the maximum value of this expression is $15,000 + 20,000\pi$. This is the maximum carrier deviation $\Delta\omega$. Hence,

$$\Delta f = \frac{\Delta\omega}{2\pi} = 12,387.32 \text{ Hz}$$

(c)

$$\beta = \frac{\Delta f}{B} = \frac{12,387.32}{1000} = 12.387$$

(e)

$$B_{EM} = 2(\Delta f + B) = 26,774.65 \text{ Hz}$$

Observe the generality of this method of estimating the bandwidth of an angle-modulated waveform. We need not know whether it is FM, PM, or some other kind of angle modulation. It is applicable to any angle-modulated signal.