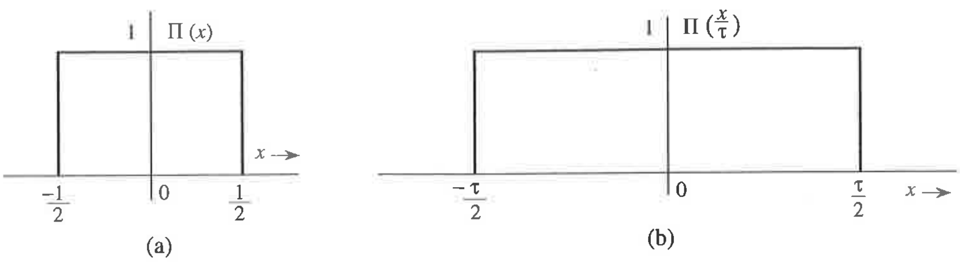

Figure 3.6 Rectangular pulse.

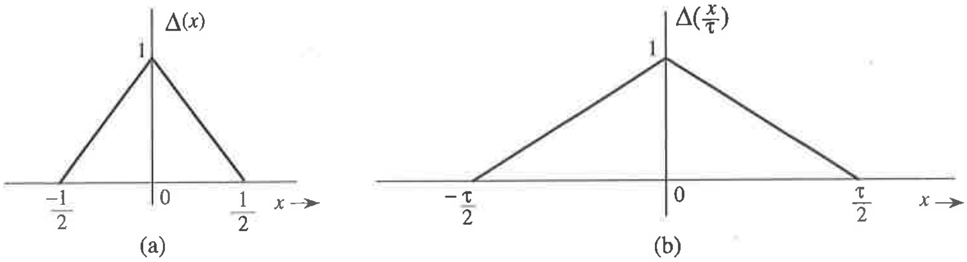

Figure 3.7 Triangular pulse.

Unit Rectangular Function

We use the pictorial notation $\Pi(x)$ for a rectangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.6a:

$$\Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0.5 & |x| = \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \tag{3.16}$$

Notice that the rectangular pulse in Fig. 3.6b is the unit rectangular pulse $\Pi(x)$ expanded by a factor τ and therefore can be expressed as $\Pi(x/\tau)$. Observe that the denominator τ in $\Pi(x/\tau)$ indicates the width of the pulse.

Unit Triangular Function

We use the pictorial notation $\Delta(x)$ for a triangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.7a:

$$\Delta(x) = \begin{cases} 1 - 2|t| & |x| < \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \tag{3.17}$$

Observe that the pulse in Fig. 3.7b is $\Delta(x/\tau)$. Observe that here, as for the rectangular pulse, the denominator τ in $\Delta(x/\tau)$ indicates the pulse width.

Sinc Function sinc(x)

The function $\sin x/x$ is the "sine over argument" function denoted by sinc (x).*

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* sinc (x) is also denoted by Sa (x) in the literature. Some authors define sinc (x) as

$$\text{sinc}\,(x) = \frac{\sin \pi x}{\pi x}$$

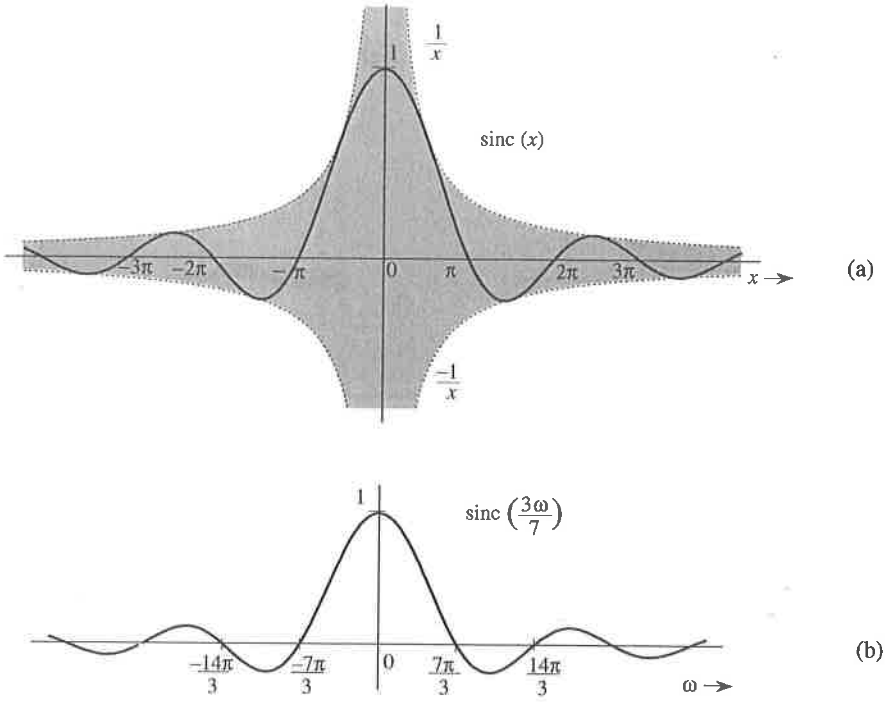


Figure 3.8
Sinc pulse.

This function plays an important role in signal processing. We define

$$\text{sinc}\,(x) = \frac{\sin x}{x} \tag{3.18}$$

Inspection of Eq. (3.18) shows that

1. sinc (x) is an even function of x.
2. sinc $(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it is indeterminate. This means that sinc $(x) = 0$ for $t = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
3. Using L'Hôspital's rule, we find sinc $(0) = 1$.
4. sinc (x) is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, sinc (x) exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
5. In summary, sinc (x) is an even oscillating function with decreasing amplitude. It has a unit peak at $x = 0$ and zero crossings at integer multiples of π.

Figure 3.8a shows sinc (x). Observe that sinc $(x) = 0$ for values of x that are positive and negative integral multiples of π. Figure 3.8b shows sinc $(3\omega/7)$. The argument $3\omega/7 = \pi$ when $\omega = 7\pi/3$ or $f = 7/6$. Therefore, the first zero of this function occurs at $\omega = 7\pi/3$ $(f = 7/6)$.

Example 3.2 Find the Fourier transform of $g(t) = \Pi(t/\tau)$ (Fig. 3.9a).

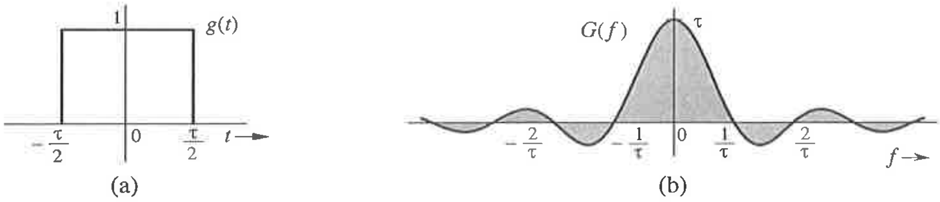


Figure 3.9 Rectangular pulse and its Fourier spectrum.

We have

$$G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi ft}\,dt$$

Since $\Pi(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$\begin{aligned} G(f) &= \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\,dt \\ &= -\frac{1}{j2\pi f}(e^{-j\pi f\tau} - e^{j\pi f\tau}) = \frac{2\sin(\pi f\tau)}{2\pi f} \\ &= \tau\frac{\sin(\pi f\tau)}{(\pi f\tau)} = \tau\ \text{sinc}\ (\pi f\tau) \end{aligned}$$

Therefore,

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\ \text{sinc}\left(\frac{\omega\tau}{2}\right) = \tau\ \text{sinc}\ (\pi f\tau) \tag{3.19}$$

Recall that $\text{sinc}\,(x) = 0$ when $x = \pm n\pi$. Hence, $\text{sinc}\,(\omega\tau/2) = 0$ when $\omega\tau/2 = \pm n\pi$; that is, when $f = \pm n/\tau$ $(n = 1, 2, 3, \ldots)$, as shown in Fig. 3.9b. Observe that in this case $G(f)$ happens to be real. Hence, we may convey the spectral information by a single plot of $G(f)$ shown in Fig. 3.9b.

Example 3.3 Find the Fourier transform of the unit impulse signal $\delta(t)$.

We use the sampling property of the impulse function [Eq. (2.11)] to obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\,dt = e^{-j2\pi f\cdot 0} = 1 \tag{3.20a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{3.20b}$$

Figure 3.10 shows $\delta(t)$ and its spectrum.

Figure 3.10 Unit impulse and its Fourier spectrum.

Example 3.4 Find the inverse Fourier transform of $\delta(2\pi f) = \frac{1}{2\pi}\delta(f)$.

From Eq. (3.9b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(2\pi f)] = \int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,d(2\pi f)$$
$$= \frac{1}{2\pi}\cdot e^{-j2\pi f\cdot 0} = \frac{1}{2\pi}$$

Therefore,

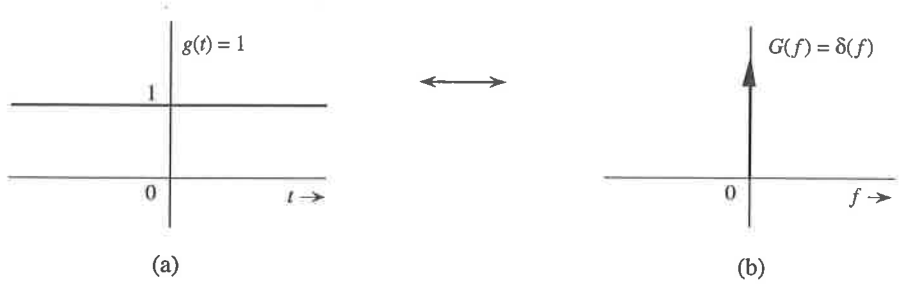
$$\frac{1}{2\pi} \Longleftrightarrow \delta(2\pi f) \tag{3.21a}$$

or

$$1 \Longleftrightarrow \delta(f) \tag{3.21b}$$

This shows that the spectrum of a constant signal $g(t) = 1$ is an impulse $\delta(f) = 2\pi\delta(2\pi f)$, as shown in Fig. 3.11.

Figure 3.11 Constant (dc) signal and its Fourier spectrum.



The result [Eq. (3.21b)] also could have been anticipated on qualitative grounds. Recall that the Fourier transform of $g(t)$ is a spectral representation of $g(t)$ in terms of everlasting exponential components of the form $e^{j2\pi ft}$. Now to represent a constant signal $g(t) = 1$, we need a single everlasting exponential $e^{j2\pi ft}$ with $f = 0$. This results in a spectrum at a single frequency $f = 0$. We could also say that $g(t) = 1$ is a dc signal that has a single frequency component at $f = 0$ (dc).

If an impulse at $f = 0$ is a spectrum of a dc signal, what does an impulse at $f = f_0$ represent? We shall answer this question in the next example.

Example 3.5 Find the inverse Fourier transform of $\delta(f - f_0)$.

We the sampling property of the impulse function to obtain

$$\mathcal{F}^{-1}[\delta(f - f_0)] = \int_{-\infty}^{\infty} \delta(f - f_0)e^{j2\pi ft}\, df = e^{j2\pi f_0 t}$$

Therefore,

$$e^{j2\pi f_0 t} \Longleftrightarrow \delta(f - f_0) \tag{3.22a}$$

This result shows that the spectrum of an everlasting exponential $e^{j2\pi f_0 t}$ is a single impulse at $f = f_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j2\pi f_0 t}$, we need a single everlasting exponential $e^{j2\pi ft}$ with $\omega = 2\pi f_0$. Therefore, the spectrum consists of a single component at frequency $f = f_0$.

From Eq. (3.22a) it follows that

$$e^{-j2\pi f_0 t} \Longleftrightarrow \delta(f + f_0) \tag{3.22b}$$

Example 3.6 Find the Fourier transforms of the everlasting sinusoid cos $2\pi f_0$.

Recall the Euler formula

$$\cos 2\pi f_0 t = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

Adding Eqs. (3.22a) and (3.22b), and using the preceding formula, we obtain

$$\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)] \tag{3.23}$$

The spectrum of cos $2\pi f_0 t$ consists of two impulses at f_0 and $-f_0$ in the f-domain, or, two impulses at $\pm\omega_0 = \pm 2\pi f_0$ in the ω-domain as shown in Fig. 3.12. The result also follows from qualitative reasoning. An everlasting sinusoid cos $\omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components of frequencies ω_0 and $-\omega_0$.

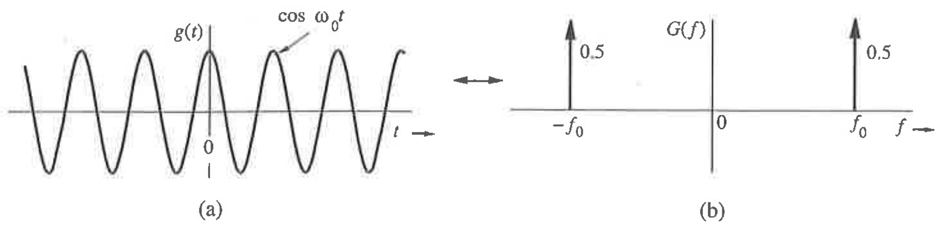


Figure 3.12 Cosine signal and its Fourier spectrum.

Example 3.7 Find the Fourier transform of the sign function sgn (t) (pronounced signum t), shown in Fig. 3.13. Its value is $+1$ or -1, depending on whether t is positive or negative:

$$\text{sgn}\,(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \tag{3.24}$$

We cannot use integration to find the transform of sgn (t) directly. This is because sgn (t) violates the Dirichlet condition [see E.g. (3.14) and the associated footnote]. Specifically, sgn (t) is not absolutely integrable. However, the transform can be obtained by considering sgn t as a sum of two exponentials, as shown in Fig. 3.13, in the limit as $a \to 0$:

$$\text{sgn}\,t = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

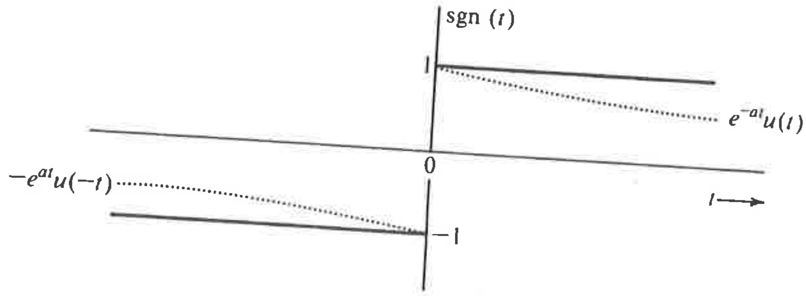


Figure 3.13 Sign function.

Therefore,

$$\begin{aligned}
\mathcal{F}\,[\text{sgn}(t)] &= \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\
&= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\
&= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f}
\end{aligned} \tag{3.25}$$

3.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

We now study some of the important properties of the Fourier transform and their implications as well as their applications. Before embarking on this study, it is important to point out a pervasive aspect of the Fourier transform—the **time-frequency duality**.

TABLE 3.1
Short Table of Fourier Transforms

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\dfrac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\dfrac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\dfrac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}\,(\pi f\tau)$	
18	$2B\,\mathrm{sinc}\,(2\pi Bt)$	$\Pi\left(\dfrac{f}{2B}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\,\mathrm{sinc}^2\left(\dfrac{\pi f\tau}{2}\right)$	
20	$B\,\mathrm{sinc}^2\,(\pi Bt)$	$\Delta\left(\dfrac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\frac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

3.3.1 Time-Frequency Duality

Equations (3.9) show an interesting fact: the direct and the inverse transform operations are remarkably similar. These operations, required to go from $g(t)$ to $G(f)$ and then from $G(f)$ to $g(t)$, are shown graphically in Fig. 3.14. The only minor difference between these two operations lies in the opposite signs used in their exponential indices.

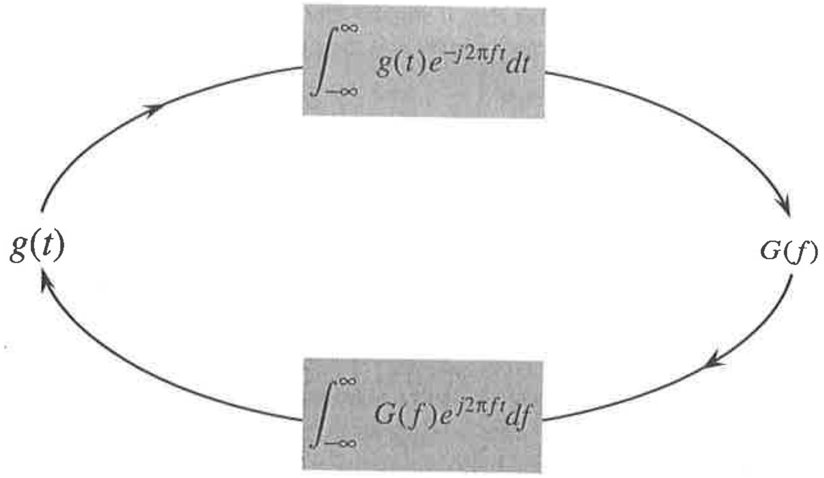


Figure 3.14 Near symmetry between direct and inverse Fourier transforms.

This similarity has far-reaching consequences in the study of Fourier transforms. It is the basis of the so-called duality of time and frequency. *The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, the negative can be obtained from the photograph.* For any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of the factor 2π and a sign change). For example, the time-shifting property, to be proved later, states that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0}$$

The dual of this property (the frequency-shifting property) states that

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f - f_0)$$

Observe the role reversal of time and frequency in these two equations (with the minor difference of the sign change in the exponential index). The value of this principle lies in the fact that *whenever we derive any result, we can be sure that it has a dual.* This knowledge can give valuable insights about many unsuspected properties or results in signal processing.

The properties of the Fourier transform are useful not only in deriving the direct and the inverse transforms of many functions, but also in obtaining several valuable results in signal processing. The reader should not fail to observe the ever-present duality in this discussion. We begin with the duality property, which is one of the consequences of the duality principle.

3.3.2 Duality Property

The duality property states that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$G(t) \Longleftrightarrow g(-f) \tag{3.26}$$

The duality property states that if the Fourier transform of $g(t)$ is $G(f)$ then the Fourier transform of $G(t)$, with f replaced by t, is the $g(-f)$ which is the original time domain signal with t replaced by $-f$.

Proof: From Eq. (3.9b),

$$g(t) = \int_{-\infty}^{\infty} G(x)e^{j2\pi xt}\, dx$$

Hence,

$$g(-t) = \int_{-\infty}^{\infty} G(x)e^{-j2\pi xt}\, dx$$

Changing t to f yields Eq. (3.26). ■

Example 3.8 In this example we shall apply the duality property [Eq. (3.26)] to the pair in Fig. 3.15a.

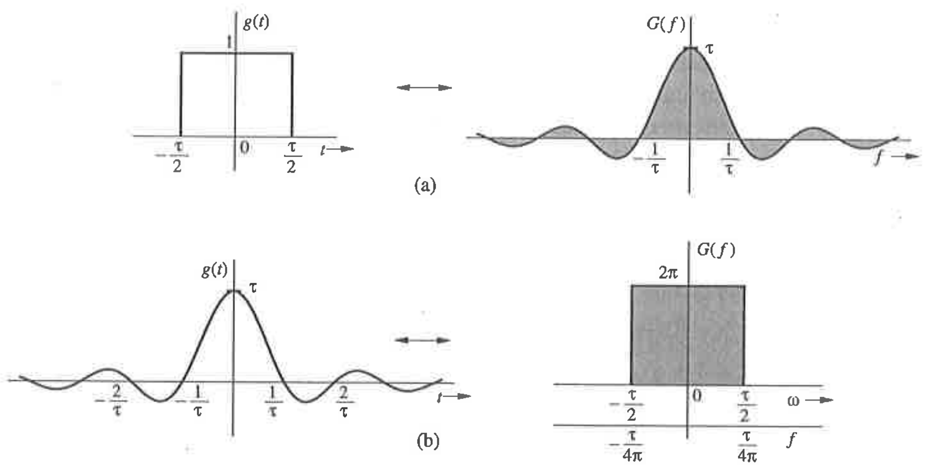


Figure 3.15 Duality property of the Fourier transform.

From Eq. (3.19) we have

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}(\pi f \tau) \tag{3.27a}$$

$$\underbrace{\Pi\left(\frac{t}{\alpha}\right)}_{g(t)} \Longleftrightarrow \underbrace{\alpha \operatorname{sinc}(\pi f \alpha)}_{G(f)} \tag{3.27b}$$

Also $G(t)$ is the same as $G(f)$ with f replaced by t, and $g(-f)$ is the same as $g(t)$ with t replaced by $-f$. Therefore, the duality property (3.26) yields

$$\underbrace{\alpha \operatorname{sinc}(\pi \alpha t)}_{G(t)} \Longleftrightarrow \underbrace{\Pi\left(-\frac{f}{\alpha}\right)}_{g(-f)} = \Pi\left(\frac{f}{\alpha}\right) \tag{3.28a}$$

Substituting $\tau = 2\pi\alpha$, we obtain

$$\tau \operatorname{sinc}\left(\frac{\alpha t}{2}\right) \Longleftrightarrow 2\pi\, \Pi\left(\frac{2\pi f}{\tau}\right) \tag{3.28b}$$

In Eq. (3.8) we used the fact that $\Pi(-t) = \Pi(t)$ because $\Pi(t)$ is an even function. Figure 3.15b shows this pair graphically. Observe the interchange of the roles of t and $2\pi f$ (with the minor adjustment of the factor 2π). This result appears as pair 18 in Table 3.1 (with $\tau/2 = W$).

As an interesting exercise, generate a dual of every pair in Table 3.1 by applying the duality property.

3.3.3 Time-Scaling Property

If

$$g(t) \Longleftrightarrow G(f)$$

then, for any real constant a,

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right) \tag{3.29}$$

Proof: For a positive real constant a,

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j2\pi ft}\, dt = \frac{1}{a}\int_{-\infty}^{\infty} g(x) e^{(-j2\pi f/a)x}\, dx = \frac{1}{a} G\left(\frac{f}{a}\right)$$

Similarly, it can be shown that if $a < 0$,

$$g(at) \Longleftrightarrow \frac{-1}{a} G\left(\frac{f}{a}\right)$$

Hence follows Eq. (3.29). ■

Significance of the Time-Scaling Property

The function $g(at)$ represents the function $g(t)$ compressed in time by a factor a ($|a| > 1$). Similarly, a function $G(f/a)$ represents the function $G(f)$ expanded in frequency by the same factor a. *The time-scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.* Intuitively, compression in time by a factor a means that the signal is varying **more rapidly** by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed. For instance, the signal $\cos 4\pi f_0 t$ is the same as the signal $\cos 2\pi f_0 t$ time-compressed by a factor of 2. Clearly, the spectrum of the former (impulse at $\pm 2f_0$) is an expanded version of the spectrum of the latter (impulse at $\pm f_0$). The effect of this scaling is demonstrated in Fig. 3.16.

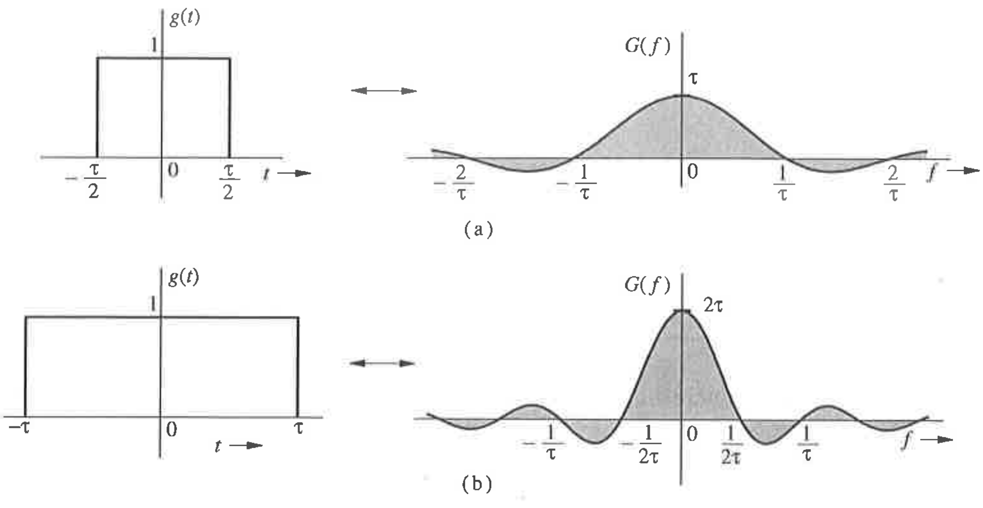


Figure 3.16 Scaling property of the Fourier transform.

Reciprocity of Signal Duration and Its Bandwidth

The time-scaling property implies that if $g(t)$ is wider, its spectrum is narrower, and vice versa. Doubling the signal duration halves its bandwidth, and vice versa. This suggests that the bandwidth of a signal is inversely proportional to the signal duration or width (in seconds). We have already verified this fact for the rectangular pulse, where we found that the bandwidth of a gate pulse of width τ seconds is $1/\tau$ Hz. More discussion of this interesting topic can be found in the literature.[2]

Example 3.9 Show that

$$g(-t) \Longleftrightarrow G(-f) \tag{3.30}$$

Use this result and the fact that $e^{-at}u(t) \Longleftrightarrow 1/(a + j2\pi f)$, to find the Fourier transforms of $e^{at}u(-t)$ and $e^{-a|t|}$.

Equation (3.30) follows from Eq. (3.29) by letting $a = -1$. Application of Eq. (3.30) to pair 1 of Table 3.1 yields

$$e^{at}u(-t) \Longleftrightarrow \frac{1}{a - j2\pi f}$$

Also

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Therefore,

$$e^{-a|t|} \Longleftrightarrow \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2} \tag{3.31}$$

The signal $e^{-a|t|}$ and its spectrum are shown in Fig. 3.17.

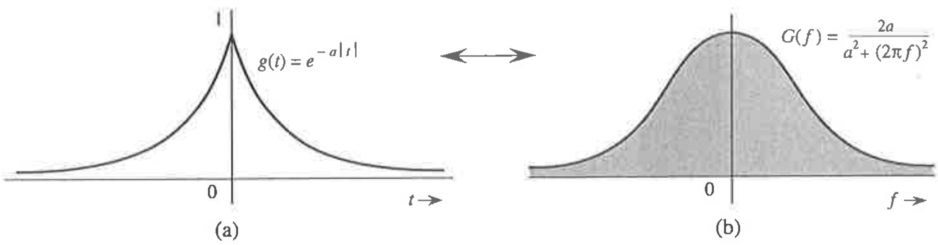


Figure 3.17 $e^{-a|t|}$ and its Fourier spectrum.

3.3.4 Time-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0} \tag{3.32a}$$

Proof: By definition,

$$\mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi f t}\, dt$$

Letting $t - t_0 = x$, we have

$$\begin{aligned}\mathcal{F}[g(t - t_0)] &= \int_{-\infty}^{\infty} g(x)e^{-j2\pi f(x+t_0)}\, dx \\ &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(x)e^{-j2\pi f x}\, dx = G(f)e^{-j2\pi f t_0}\end{aligned} \tag{3.32b}$$

This result shows that *delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-2\pi f t_0$.*

Physical Explanation of the Linear Phase

Time delay in a signal causes a linear phase shift in its spectrum. This result can also be derived by heuristic reasoning. Imagine $g(t)$ being synthesized by its Fourier components, which are sinusoids of certain amplitudes and phases. The delayed signal $g(t - t_0)$ can be synthesized by the same sinusoidal components, each delayed by t_0 seconds. The amplitudes of the components remain unchanged. Therefore, the amplitude spectrum of $g(t - t_0)$ is identical to that of $g(t)$. The time delay of t_0 in each sinusoid, however, does change the phase of each component. Now, a sinusoid $\cos 2\pi f t$ delayed by t_0 is given by

$$\cos 2\pi f(t - t_0) = \cos (2\pi f t - 2\pi f t_0)$$

Therefore, a time delay t_0 in a sinusoid of frequency f manifests as a phase delay of $2\pi f t_0$. This is a linear function of f, meaning that higher frequency components must undergo proportionately

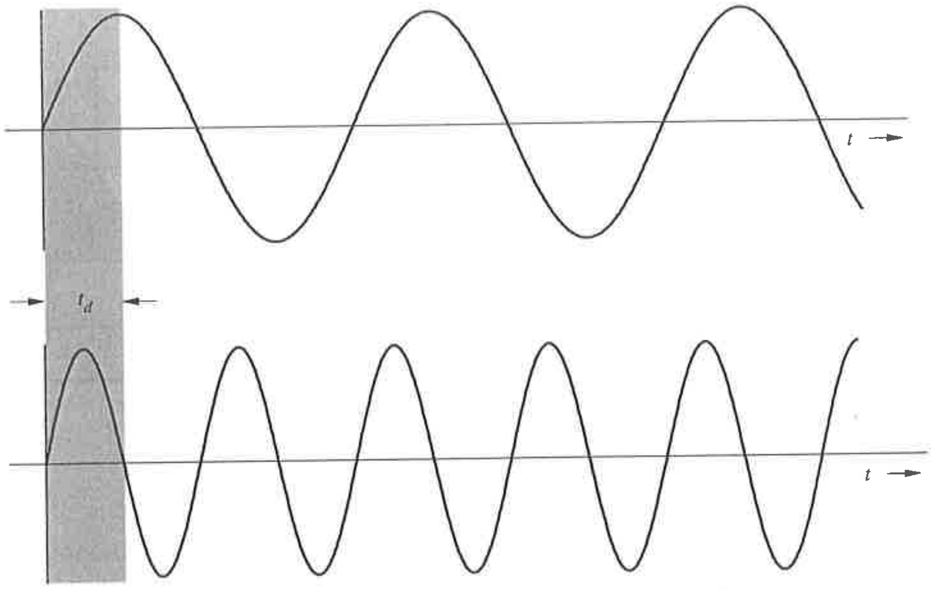


Figure 3.18 Physical explanation of the time-shifting property.

higher phase shifts to achieve the same time delay. This effect is shown in Fig. 3.18 with two sinusoids, the frequency of the lower sinusoid being twice that of the upper. The same time delay t_0 amounts to a phase shift of $\pi/2$ in the upper sinusoid and a phase shift of π in the lower sinusoid. This verifies that *to achieve the same time delay, higher frequency sinusoids must undergo proportionately higher phase shifts.*

Example 3.10 Find the Fourier transform of $e^{-a|t-t_0|}$.

This function, shown in Fig. 3.19a, is a time-shifted version of $e^{-a|t|}$ (shown in Fig. 3.17a). From Eqs. (3.31) and (3.32) we have

$$e^{-a|t-t_0|} \Longleftrightarrow \frac{2a}{a^2+(2\pi f)^2}e^{-j2\pi f t_0} \tag{3.33}$$

The spectrum of $e^{-a|t-t_0|}$ (Fig. 3.19b) is the same as that of $e^{-a|t|}$ (Fig. 3.17b), except for an added phase shift of $-2\pi f t_0$.

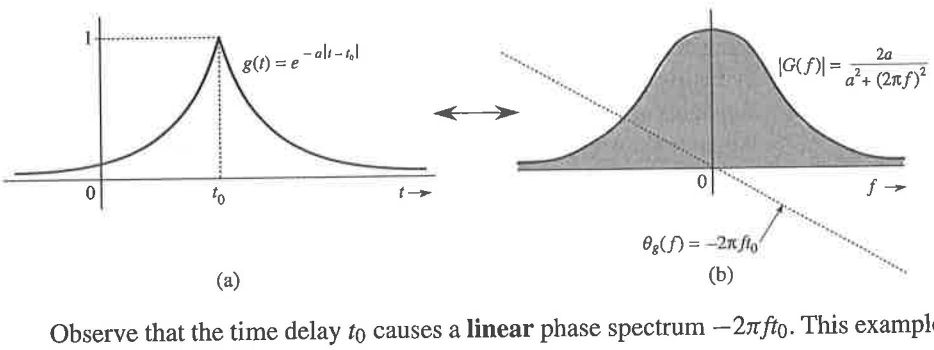


Figure 3.19 Effect of time shifting on the Fourier spectrum of a signal.

Observe that the time delay t_0 causes a **linear** phase spectrum $-2\pi f t_0$. This example clearly demonstrates the effect of time shift.

3.3.5 Frequency-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f - f_0) \tag{3.34}$$

This property is also called the modulation property.

Proof: By definition,

$$\mathcal{F}[g(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j2\pi f_0 t}e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} g(t)e^{-j(2\pi f - 2\pi f_0)t}\,dt = G(f - f_0)$$

■

This property states that multiplication of a signal by a factor $e^{j2\pi f_0 t}$ shifts the spectrum of that signal by $f = f_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Changing f_0 to $-f_0$ in Eq. (3.34) yields

$$g(t)e^{-j2\pi f_0 t} \Longleftrightarrow G(f + f_0) \tag{3.35}$$

Because $e^{j2\pi f_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $g(t)$ by a sinusoid. This can be seen from

$$g(t)\cos 2\pi f_0 t = \frac{1}{2}\left[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}\right]$$

From Eqs. (3.34) and (3.35), it follows that

$$g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f - f_0) + G(f + f_0)] \tag{3.36}$$

This shows that the multiplication of a signal $g(t)$ by a sinusoid of frequency f_0 shifts the spectrum $G(f)$ by $\pm f_0$. Multiplication of a sinusoid $\cos 2\pi f_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. This type of modulation is known as **amplitude modulation**. The sinusoid $\cos 2\pi f_0 t$ is called the **carrier**, the signal $g(t)$ is the **modulating signal**, and the signal $g(t)\cos 2\pi f_0 t$ is the **modulated signal**. Modulation and demodulation will be discussed in detail in Chapters 4 and 5.

To sketch a signal $g(t)\cos 2\pi f_0 t$, we observe that

$$g(t)\cos 2\pi f_0 t = \begin{cases} g(t) & \text{when } \cos 2\pi f_0 t = 1 \\ -g(t) & \text{when } \cos 2\pi f_0 t = -1 \end{cases}$$

Therefore, $g(t)\cos 2\pi f_0 t$ touches $g(t)$ when the sinusoid $\cos 2\pi f_0 t$ is at its positive peaks and touches $-g(t)$ when $\cos 2\pi f_0 t$ is at its negative peaks. This means that $g(t)$ and $-g(t)$ act as envelopes for the signal $g(t)\cos 2\pi f_0 t$ (see Fig. 3.20c). The signal $-g(t)$ is a mirror image of $g(t)$ about the horizontal axis. Figure 3.20 shows the signals $g(t)$, $g(t)\cos 2\pi f_0 t$, and their respective spectra.

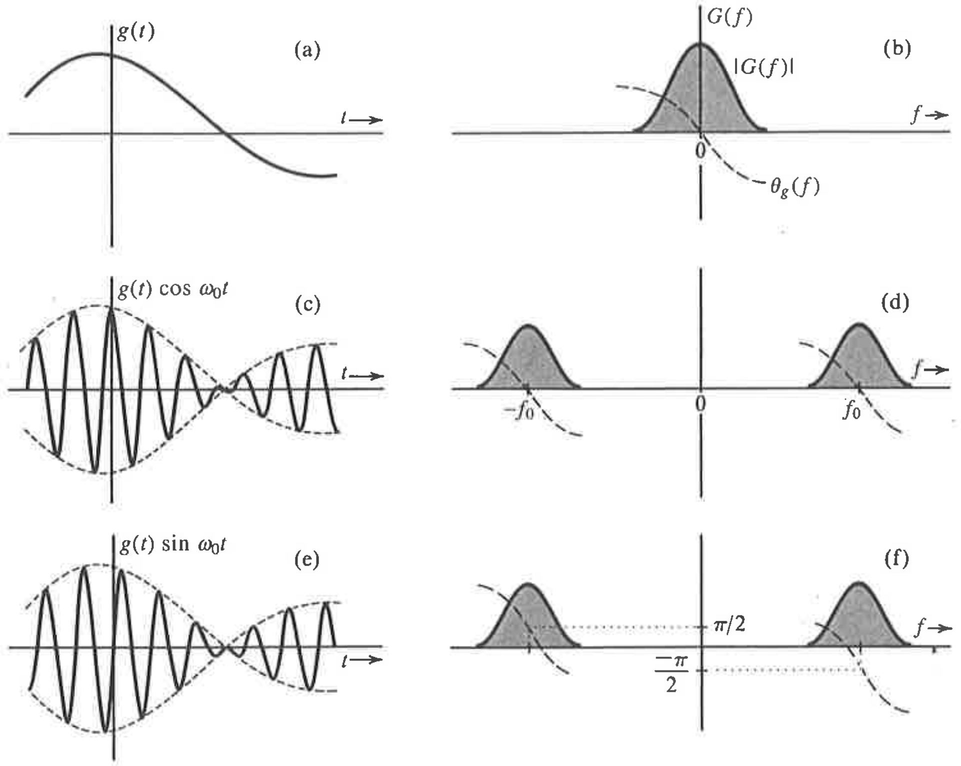


Figure 3.20 Amplitude modulation of a signal causes spectral shifting.

Shifting the Phase Spectrum of a Modulated Signal

We can shift the phase of each spectral component of a modulated signal by a constant amount θ_0 merely by using a carrier $\cos(2\pi f_0 t + \theta_0)$ instead of $\cos 2\pi f_0 t$. If a signal $g(t)$ is multiplied by $\cos(2\pi f_0 t + \theta_0)$, then we can use an argument similar to that used to derive Eq. (3.36), to show that

$$g(t)\cos(2\pi f_0 t + \theta_0) \Longleftrightarrow \frac{1}{2}\left[G(f - f_0)\, e^{j\theta_0} + G(f + f_0)\, e^{-j\theta_0}\right] \tag{3.37}$$

For a special case when $\theta_0 = -\pi/2$, Eq. (3.37) becomes

$$g(t)\sin 2\pi f_0 t \Longleftrightarrow \frac{1}{2}\left[G(f - f_0)\, e^{-j\pi/2} + G(f + f_0)\, e^{j\pi/2}\right] \tag{3.38}$$

Observe that $\sin 2\pi f_0 t$ is $\cos 2\pi f_0 t$ with a phase delay of $\pi/2$. Thus, shifting the carrier phase by $\pi/2$ shifts the phase of every spectral component by $\pi/2$. Figures 3.20e and f show the signal $g(t)\sin 2\pi f_0 t$ and its spectrum.

Modulation is a common application that shifts signal spectra. In particular, If several message signals, each occupying the same frequency band, are transmitted simultaneously over a common transmission medium, they will all interfere; it will be impossible to separate or retrieve them at a receiver. For example, if all radio stations decide to broadcast audio signals simultaneously, receivers will not be able to separate them. This problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band, which is not occupied by any other station. A radio receiver can pick up any station by tuning to the

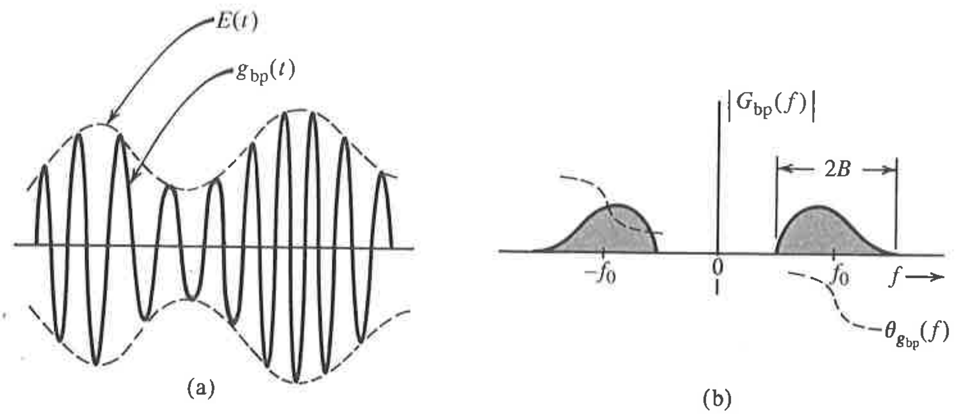


Figure 3.21 Bandpass signal and its spectrum.

band of the desired station. The receiver must now demodulate the received signal (undo the effect of modulation). Demodulation therefore consists of another spectral shift required to restore the signal to its original band.

Bandpass Signals

Figure 3.20(d)(f) shows that if $g_c(t)$ and $g_s(t)$ are low-pass signals, each with a bandwidth B Hz or $2\pi B$ rad/s, then the signals $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are both bandpass signals occupying the same band, and each having a bandwidth of $2B$ Hz. Hence, a linear combination of both these signals will also be a bandpass signal occupying the same band as that of the either signal, and with the same bandwidth ($2B$ Hz). Hence, a general bandpass signal $g_{bp}(t)$ can be expressed as*

$$g_{bp}(t) = g_c(t)\cos 2\pi f_0 t + g_s(t)\sin 2\pi f_0 t \tag{3.39}$$

The spectrum of $g_{bp}(t)$ is centered at $\pm f_0$ and has a bandwidth $2B$, as shown in Fig. 3.21. Although the magnitude spectra of both $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are symmetrical about $\pm f_0$, the magnitude spectrum of their sum, $g_{bp}(t)$, is not necessarily symmetrical about $\pm f_0$. This is because the different phases of the two signals do not allow their amplitudes to add directly for the reason that

$$a_1 e^{j\varphi_1} + a_2 e^{j\varphi_2} \neq (a_1 + a_2)e^{j(\varphi_1+\varphi_2)}$$

A typical bandpass signal $g_{bp}(t)$ and its spectra are shown in Fig. 3.21. We can use a well-known trigonometric identity to express Eq. (3.39) as

$$g_{bp}(t) = E(t)\cos[2\pi f_0 t + \psi(t)] \tag{3.40}$$

where

$$E(t) = +\sqrt{g_c^2(t) + g_s^2(t)} \tag{3.41a}$$

$$\psi(t) = -\tan^{-1}\left[\frac{g_s(t)}{g_c(t)}\right] \tag{3.41b}$$

* See Sec. 9.9 for a rigorous proof of this statement.

Because $g_c(t)$ and $g_s(t)$ are low-pass signals, $E(t)$ and $\psi(t)$ are also low-pass signals. Because $E(t)$ is nonnegative [Eq. (3.41a)], it follows from Eq. (3.40) that $E(t)$ is a slowly varying envelope and $\psi(t)$ is a slowly varying phase of the bandpass signal $g_{bp}(t)$, as shown in Fig. 3.21. Thus, the bandpass signal $g_{bp}(t)$ will appear as a sinusoid of slowly varying amplitude. Because of the time-varying phase $\psi(t)$, the frequency of the sinusoid also varies slowly* with time about the center frequency f_0.

Example 3.11 Find the Fourier transform of a general periodic signal $g(t)$ of period T_0, and hence, determine the Fourier transform of the periodic impulse train $\delta_{T_0}(t)$ shown in Fig. 3.22a.

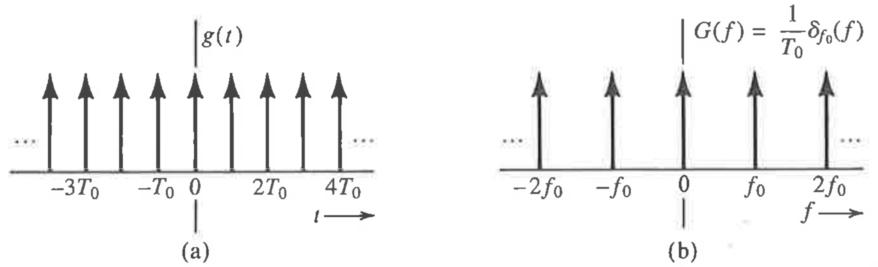


Figure 3.22 Impulse train and its spectrum.

A periodic signal $g(t)$ can be expressed as an exponential Fourier series as

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

Therefore,

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \mathcal{F}[D_n e^{jn2\pi f_0 t}]$$

Now from Eq. (3.22a), it follows that

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} D_n \delta(f - nf_0) \tag{3.42}$$

Equation (2.67) shows that the impulse train $\delta_{T_0}(t)$ can be expressed as an exponential Fourier series as

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

* It is necessary that $B \ll f_0$ for a well-defined envelope. Otherwise the variations of $E(t)$ are of the same order as the carrier, and it will be difficult to separate the envelope from the carrier.

Here $D_n = 1/T_0$. Therefore, from Eq. (3.42),

$$\delta_{T_0}(t) \iff \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$
$$= \frac{1}{T_0} \delta_{f_0}(f) \qquad f_0 = \frac{1}{T_0} \tag{3.43}$$

Thus, the spectrum of the impulse train also happens to be an impulse train (in the frequency domain), as shown in Fig. 3.23b.

3.3.6 Convolution Theorem

The convolution of two functions $g(t)$ and $w(t)$, denoted by $g(t) * w(t)$, is defined by the integral

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t - \tau)\, d\tau$$

The time convolution property and its dual, the frequency convolution property, state that if

$$g_1(t) \iff G_1(f) \qquad \text{and} \qquad g_2(t) \iff G_2(f)$$

then (**time convolution**)

$$g_1(t) * g_2(t) \iff G_1(f)G_2(f) \tag{3.44}$$

and (**frequency convolution**)

$$g_1(t)g_2(t) \iff G_1(f) * G_2(f) \tag{3.45}$$

These two relationships of the convolution theorem state that convolution of two signals in the time domain becomes multiplication in the frequency domain, while multiplication of two signals in the time domain becomes convolution in the frequency domain.

Proof: By definition,

$$\mathcal{F}|g_1(t) * g_2(t)| = \int_{-\infty}^{\infty} e^{-j2\pi ft} \left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau \right] dt$$
$$= \int_{-\infty}^{\infty} g_1(\tau) \left[\int_{-\infty}^{\infty} e^{-j2\pi ft} g_2(t - \tau)dt \right] d\tau$$

The inner integral is the Fourier transform of $g_2(t - \tau)$, given by [time-shifting property in Eq. (3.32a)] $G_2(f)e^{-j2\pi f\tau}$. Hence,

$$\mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau} G_2(f)\, d\tau$$
$$= G_2(f) \int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau}\, d\tau = G_1(f)G_2(f)$$

■

The frequency convolution property (3.45) can be proved in exactly the same way by reversing the roles of $g(t)$ and $G(f)$.

Bandwidth of the Product of Two Signals

If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz. This result follows from the application of the width property of convolution[3] to Eq. (3.45). This property states that the width of $x * y$ is the sum of the widths of x and y. Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz.*

Example 3.12 Using the time convolution property, show that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.46}$$

Because

$$u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$$

it follows that

$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} g(\tau)\,d\tau$$

Now from the time convolution property [Eq. (3.44)], it follows that

$$\begin{aligned} g(t) * u(t) \Longleftrightarrow\ & G(f)U(f) \\ =& G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ =& \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \end{aligned}$$

In deriving the last result we used pair 11 of Table 3.1 and Eq. (2.10a).

3.3.7 Time Differentiation and Time Integration

If

$$g(t) \Longleftrightarrow G(f),$$

---

* The width property of convolution does not hold in some pathological cases. It fails when the convolution of two functions is zero over a range even when both functions are nonzero [e.g., $\sin 2\pi f_0 t\, u(t) * u(t)$]. Technically the property holds even in this case if in calculating the width of the convolved function, we take into account the range in which the convolution is zero.

then **(time differentiation)***

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi f G(f) \tag{3.47}$$

and **(time integration)**

$$\int_{-\infty}^{t} g(\tau)\,d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.48}$$

Proof: Differentiation of both sides of Eq. (3.9b) yields

$$\frac{dg(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f G(f) e^{j2\pi f t}\, df$$

This shows that

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi f G(f)$$

Repeated application of this property yields

$$\frac{d^n g(t)}{dt^n} \Longleftrightarrow (j2\pi f)^n G(f) \tag{3.49}$$

The time integration property [Eq. (3.48)] already has been proved in Example 3.12. ■

Example 3.13 Use the time differentiation property to find the Fourier transform of the triangular pulse $\Delta(t/\tau)$ shown in Fig. 3.23a.

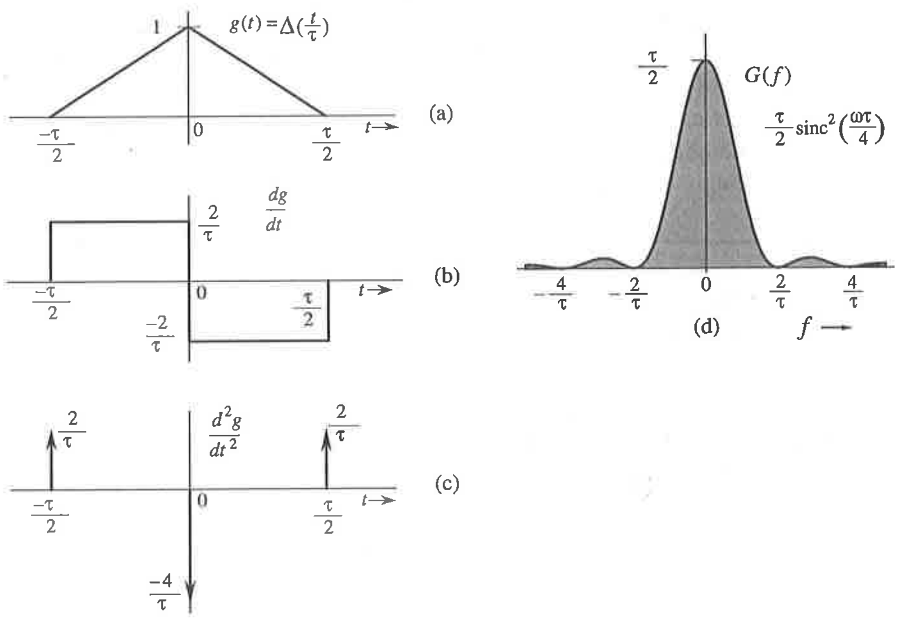


Figure 3.23 Using the time differentiation property to finding the Fourier transform of a piecewise-linear signal.

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* Valid only if the transform of $dg(t)/dt$ exists.

To find the Fourier transform of this pulse, we differentiate it successively, as shown in Fig. 3.23b and c. The second derivative consists of a sequence of impulses (Fig. 3.23c). Recall that the derivative of a signal at a jump discontinuity is an impulse of strength equal to the amount of jump. The function $dg(t)/dt$ has a positive jump of $2/\tau$ at $t = \pm\tau/2$, and a negative jump of $4/\tau$ at $t = 0$. Therefore,

$$\frac{d^2g(t)}{dt^2} = \frac{2}{\tau}\left[\delta\left(t+\frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t-\frac{\tau}{2}\right)\right] \tag{3.50}$$

From the time differentiation property [Eq. (3.49)],

$$\frac{d^2g}{dt^2} \Longleftrightarrow (j2\pi f)^2 G(f) = -(2\pi f)^2 G(f) \tag{3.51a}$$

Also, from the time-shifting property [Eqs. (3.32)],

$$\delta(t - t_0) \Longleftrightarrow e^{-j2\pi f t_0} \tag{3.51b}$$

Taking the Fourier transform of Eq. (3.50) and using the results in Eq. (3.51), we obtain

$$(j2\pi f)^2 G(f) = \frac{2}{\tau}\left(e^{j\pi f\tau} - 2 + e^{-j\pi f\tau}\right) = \frac{4}{\tau}(\cos \pi f\tau - 1) = -\frac{8}{\tau}\sin^2\left(\frac{\pi f\tau}{2}\right)$$

and

$$G(f) = \frac{8}{(2\pi f)^2\tau}\sin^2\left(\frac{\pi f\tau}{2}\right) = \frac{\tau}{2}\left[\frac{\sin(\pi f\tau/2)}{\pi f\tau/2}\right]^2 = \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\pi f\tau}{2}\right) \tag{3.52}$$

The spectrum $G(f)$ is shown in Fig. 3.23d. This procedure of finding the Fourier transform can be applied to any function $g(t)$ made up of straight-line segments with $g(t) \to 0$ as $|t| \to \infty$. The second derivative of such a signal yields a sequence of impulses whose Fourier transform can be found by inspection. This example suggests a numerical method of finding the Fourier transform of an arbitrary signal $g(t)$ by approximating the signal by straight-line segments.

To provide easy reference, several important properties of Fourier transform are summarized in Table 3.2.

3.4 SIGNAL TRANSMISSION THROUGH A LINEAR SYSTEM

A linear time-invariant (LTI) continuous time system can be characterized equally well in either the time domain or the frequency domain. The LTI system model, illustrated in Fig. 3.24, can often be used to characterize communication channels. In communication systems and in signal processing, we are interested only in bounded-input–bounded-output (BIBO) stable linear systems. Detailed discussions on system stability can be found in the textbook by Lathi.[3]

TABLE 3.2
Properties of Fourier Transform Operations

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\|a\|} G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f - f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$

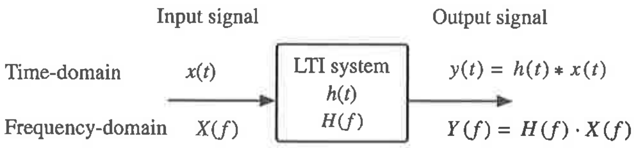

Figure 3.24 Signal transmission through a linear time-invariant system.

A stable LTI system can be characterized in the time domain by its impulse response $h(t)$, which is the system response to a unit impulse input, that is,

$$y(t) = h(t) \qquad \text{when} \qquad x(t) = \delta(t)$$

The system response to a bounded input signal $x(t)$ follows the convolutional relationship

$$y(t) = h(t) * x(t) \tag{3.53}$$

The frequency domain relationship between the input and the output is obtained by taking Fourier transform of both sides of Eq. (3.53). We let

$$x(t) \Longleftrightarrow X(f)$$
$$y(t) \Longleftrightarrow Y(f)$$
$$h(t) \Longleftrightarrow H(f)$$

Then according to the convolution theorem, Eq. (3.53) becomes

$$Y(f) = H(f) \cdot X(f) \tag{3.54}$$

Generally $H(f)$, the Fourier transform of the impulse response $h(t)$, is referred to as the **transfer function** or the **frequency response** of the LTI system. Again, in general, $H(f)$ is complex and can be written as

$$H(f) = |H(f)|e^{j\theta_h(f)}$$

where $|H(f)|$ is the amplitude response and $\theta_h(f)$ is the phase response of the LTI system.

3.4.1 Signal Distortion during Transmission

The transmission of an input signal $x(t)$ through a system changes it into the output signal $y(t)$. Equation (3.54) shows the nature of this change or modification. Here $X(f)$ and $Y(f)$ are the spectra of the input and the output, respectively. Therefore, $H(f)$ is the spectral response of the system. The output spectrum is given by the input spectrum multiplied by the spectral response of the system. Equation (3.54) clearly brings out the spectral shaping (or modification) of the signal by the system. Equation (3.54) can be expressed in polar form as

$$|Y(f)|e^{j\theta_y(f)} = |X(f)||H(f)|e^{j[\theta_x(f)+\theta_h(f)]}$$

Therefore, we have the amplitude and phase relationships

$$|Y(f)| = |X(f)|\,|H(f)| \tag{3.55a}$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f) \tag{3.55b}$$

During the transmission, the input signal amplitude spectrum $|X(f)|$ is changed to $|X(f)| \cdot |H(f)|$. Similarly, the input signal phase spectrum $\theta_x(f)$ is changed to $\theta_x(f) + \theta_h(f)$.

An input signal spectral component of frequency f is modified in amplitude by a factor $|H(f)|$ and is shifted in phase by an angle $\theta_h(f)$. Clearly, $|H(f)|$ is the amplitude response, and $\theta_h(f)$ is the phase response of the system. The plots of $|H(f)|$ and $\theta_h(f)$ as functions of f show at a glance how the system modifies the amplitudes and phases of various sinusoidal inputs. This is why $H(f)$ is called the **frequency response** of the system. During transmission through the system, some frequency components may be boosted in amplitude, while others may be attenuated. The relative phases of the various components also change. In general, the output waveform will be different from the input waveform.

3.4.2 Distortionless Transmission

In several applications, such as signal amplification or message signal transmission over a communication channel, we require the output waveform to be a replica of the input waveform. In such cases, we need to minimize the distortion caused by the amplifier or the communication channel. It is therefore of practical interest to determine the characteristics of a system that allows a signal to pass without distortion (**distortionless transmission**).

Transmission is said to be distortionless if the input and the output have identical wave shapes within a multiplicative constant. A delayed output that retains the input waveform is also considered distortionless. Thus, in distortionless transmission, the input $x(t)$ and the output $y(t)$ satisfy the condition

$$y(t) = k \cdot x(t - t_d) \tag{3.56}$$

The Fourier transform of this equation yields

$$Y(f) = kX(f)e^{-j2\pi f t_d}$$