University of Bahrain

Department of Electrical and Electronics Engineering

EENG372

Communication Systems I

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Topic 3:

PSD and Autocorrelation

This Topic will cover

- Energy and Power signals
- Energy Spectral Density
- Power Spectral Density
- Correlation
- Noise

Classification of signals

- Energy and power signals
  - A signal is an energy signal if, and only if, it has **nonzero but finite energy for all time:**

$$E_x = \lim_{T\to\infty} \int_{T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$(0 < E_x < \infty)$$

  - A signal is a power signal if, and only if, it has **finite but nonzero power for all time**:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} |x(t)|^2 dt$$

$$(0 < P_x < \infty)$$

General rule:

- **Periodic and random signals** = **power signals.**
- **deterministic and non-periodic** = **energy signals**.

1.3 SPECTRAL DENSITY

The *spectral density* of a signal characterizes the distribution of the signal's energy or power in the frequency domain. This concept is particularly important when considering filtering in communication systems. We need to be able to evaluate the signal and noise at the filter output. The energy spectral density (ESD) or the power spectral density (PSD) is used in the evaluation.

Energy Spectral Density

1.3.1 Energy Spectral Density

The total energy of a real-valued energy signal $x(t)$, defined over the interval, $(-\infty, \infty)$, is described by Equation (1.7). Using Parseval's theorem [1], we can relate the energy of such a signal expressed in the time domain to the energy expressed in the frequency domain, as

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df \tag{1.13}$$

where $X(f)$ is the Fourier transform of the nonperiodic signal $x(t)$. (For a review of Fourier techniques, see Appendix A.) Let $\psi_x(f)$ denote the squared magnitude spectrum, defined as

$$\psi_x(f) = |X(f)|^2 \tag{1.14}$$

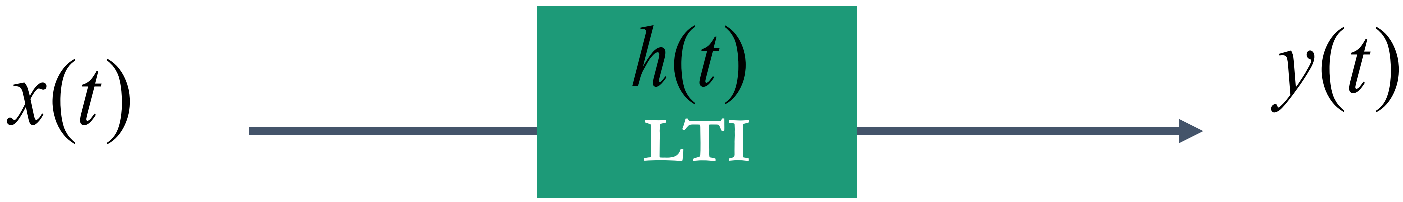
The quantify $\psi_x(f)$ is the waveform *energy spectral density* (ESD) of the signal $x(t)$. Therefore, from Equation (1.13), we can express the total energy of $x(t)$ by integrating the spectral density with respect to frequency:

$$E_x = \int_{-\infty}^{\infty} \psi_x(f)\, df \tag{1.15}$$

Energy Spectral Density (ESD)

Q: What is the ESD of the output of an LTI system?

If the input to the system is x(t):



The output is: $$y(t) = x(t) * h(t)$$

The FT of the output is: $$Y(f) = X(f)H(f)$$

The ESD of the output is:

$$|Y(f)|^2 = |X(f)H(f)|^2 = |X(f)|^2 |H(f)|^2$$

It is the input ESD multiplied by the square of the transfer function

Power Spectral Density

1.3.2 Power Spectral Density

The average power P_x of a real-valued power signal $x(t)$ is defined in Equation (1.8). If $x(t)$ is a *periodic signal* with period T_0, it is classified as a power signal. The expression for the average power of a periodic signal takes the form of Equation (1.6), where the time average is taken over the signal period T_0, as follows:

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt \qquad (1.17a)$$

Parseval's theorem for a real-valued periodic signal [1] takes the form

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad (1.17b)$$

where the $|c_n|$ terms are the complex Fourier series coefficients of the periodic signal. (See Appendix A.)

To apply Equation (1.17b), we need only know the magnitude of the coefficients, $|c_n|$. The *power spectral density* (PSD) function $G_x(f)$ of the periodic signal $x(t)$ is a real, even, and nonnegative function of frequency that gives the distribution of the power of $x(t)$ in the frequency domain, defined as

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0) \tag{1.18}$$

Equation (1.18) defines the power spectral density of a periodic signal $x(t)$ as a succession of the weighted delta functions. Therefore, the PSD of a periodic signal is a discrete function of frequency. Using the PSD defined in Equation (1.18), we can now write the average normalized power of a real-valued signal as

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = 2 \int_{0}^{\infty} G_x(f)\, df \tag{1.19}$$

Equation (1.18) describes the PSD of periodic (power) signals only. If $x(t)$ is a nonperiodic signal it *cannot* be expressed by a Fourier series, and if it is a nonperiodic power signal (having infinite energy) it *may not* have a Fourier transform. However, we may still express the power spectral density of such signals in the *limiting sense*. If we form a *truncated version* $x_T(t)$ of the nonperiodic power signal $x(t)$ by observing it only in the interval $(-T/2, T/2)$, then $x_T(t)$ has finite energy and has a proper Fourier transform $X_T(f)$. It can be shown [2] that the power spectral density of the nonperiodic $x(t)$ can then be defined in the limit as

$$G_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2 \qquad (1.20)$$

Example 1.1 Average Normalized Power

(a) Find the average normalized power in the waveform, $x(t) = A \cos 2\pi f_0 t$, using time averaging.

(b) Repeat part (a) using the summation of spectral coefficients.

Solution

(a) Using Equation (1.17a), we have

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$$

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A^2 \cos^2 2\pi f_0 t \; dt$$

$$= \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} (1 + \cos 4\pi f_0 t)\, dt$$

$$= \frac{A^2}{2T_0} (T_0) = \frac{A^2}{2}$$

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = 2 \int_{0}^{\infty} G_x(f)\, df$$

(b) Using Equations (1.18) and (1.19) gives us

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

$$\left.\begin{aligned} c_1 &= c_{-1} = \frac{A}{2} \\ c_n &= 0 \qquad \text{for } n = 0, \pm 2, \pm 3, \ldots \end{aligned}\right\} \quad \text{(see Appendix A)}$$

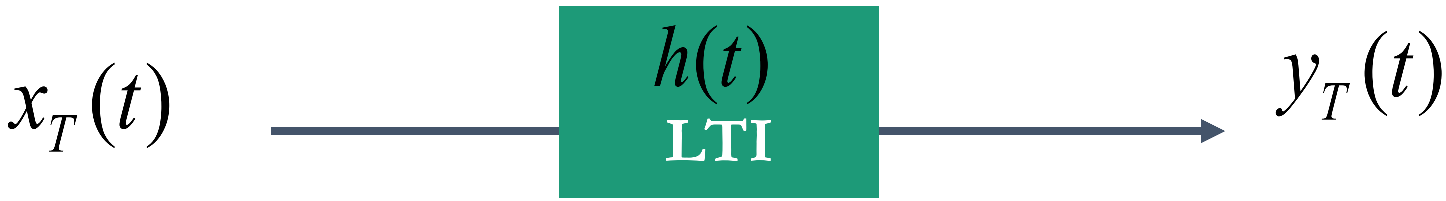
$$G_x(f) = \left(\frac{A}{2}\right)^2 \delta(f - f_0) + \left(\frac{A}{2}\right)^2 \delta(f + f_0)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = \frac{A^2}{2}$$

Power Spectral Density (PSD)

Q: What is the PSD of the output of an LTI system?

If the input to the system is x(t):



The output is: $y_T(t) = x_T(t) * h(t)$

The FT of the output is: $Y_T(f) = X_T(f)H(f)$

The PSD of the output is:

$$\lim_{T\to\infty} \frac{\left|Y_T(f)\right|^2}{T} = \lim_{T\to\infty} \frac{\left|X_T(f)H(f)\right|^2}{T} = \lim_{T\to\infty} \frac{\left|X_T(f)\right|^2}{T} \left|H(f)\right|^2$$

It is the input PSD multiplied by the square of the transfer function

Auto-correlation

1.4 AUTOCORRELATION

1.4.1 Autocorrelation of an Energy Signal

Correlation is a matching process; *autocorrelation* refers to the matching of a signal with a delayed version of itself. The autocorrelation function of a real-valued energy signal $x(t)$ is defined as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t + \tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.21}$$

The autocorrelation function $R_x(\tau)$ provides a measure of how closely the signal matches a copy of itself as the copy is shifted τ units in time. The variable τ plays the role of a scanning or searching parameter. $R_x(\tau)$ is not a function of time; it is only a function of the time difference τ between the waveform and its shifted copy.

The autocorrelative function of a real-valued *energy* signal has the following properties:

1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $R_x(\tau) \leq R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow \psi_x(f)$	autocorrelation and ESD form a Fourier transform pair, as designated by the double-headed arrows
4. $R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt$	value at the origin is equal to the energy of the signal

If items 1 through 3 are satisfied, $R_x(\tau)$ satisfies the properties of an autocorrelation function. Property 4 can be derived from property 3 and thus need not be included as a basic test.

1.4.2 Autocorrelation of a Periodic (Power) Signal

The autocorrelation function of a real-valued power signal $x(t)$ is defined as

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t + \tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.22}$$

When the power signal $x(t)$ is periodic with period T_0, the time average in Equation (1.22) may be taken over a *single period* T_0, and the autocorrelation function can be expressed as

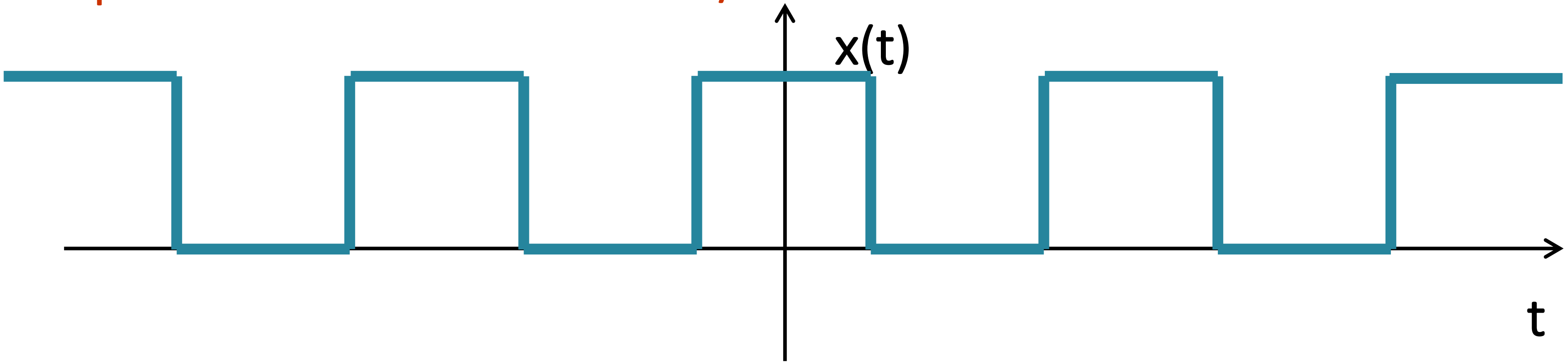
$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t + \tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.23}$$

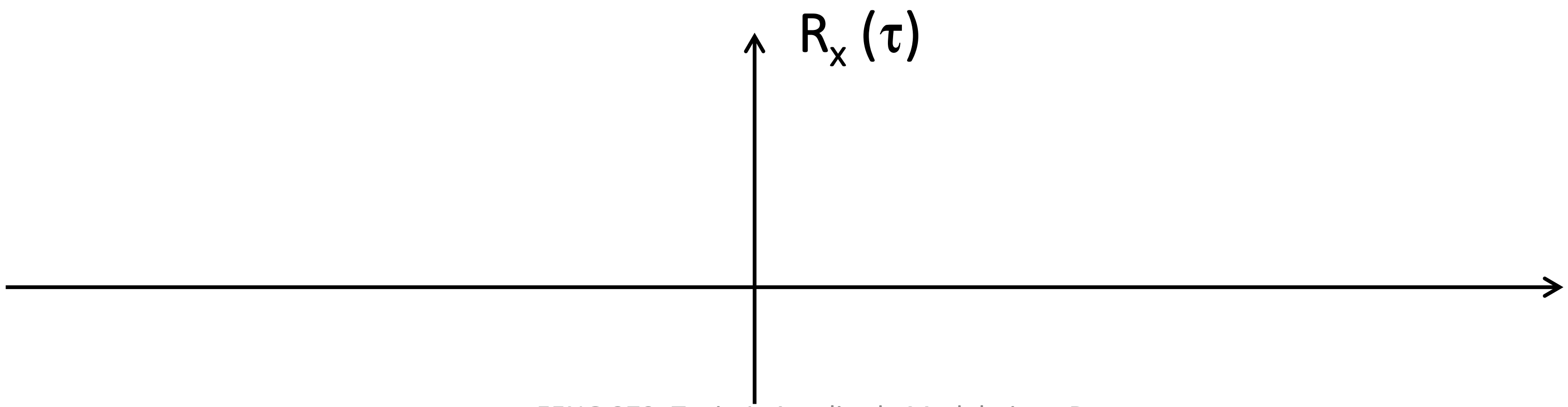
The autocorrelation function of a real-valued *periodic* signal has properties similar to those of an energy signal:

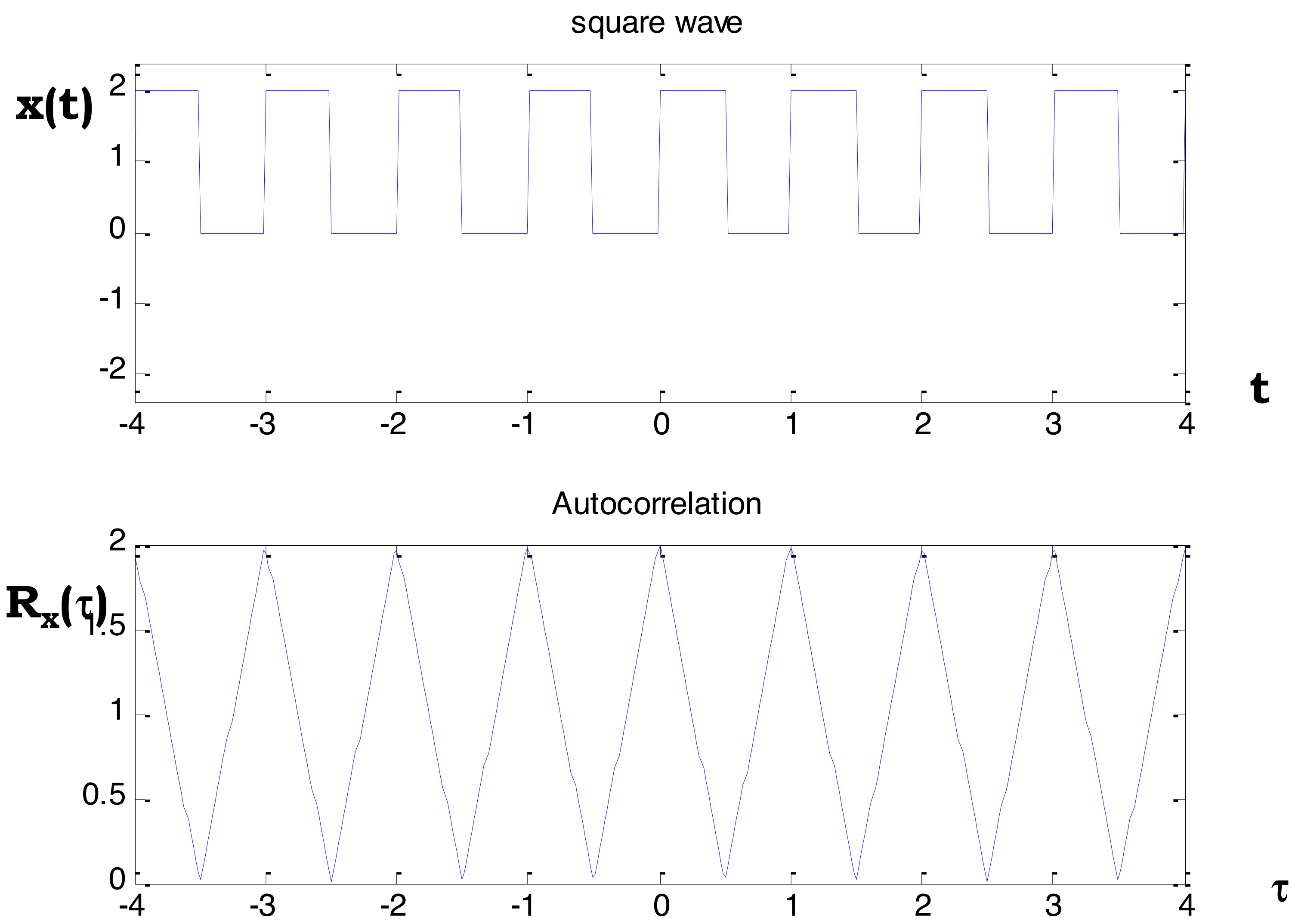
1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $R_x(\tau) \le R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow G_x(f)$	autocorrelation and PSD form a Fourier transform pair
4. $R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$	value at the origin is equal to the average power of the signal

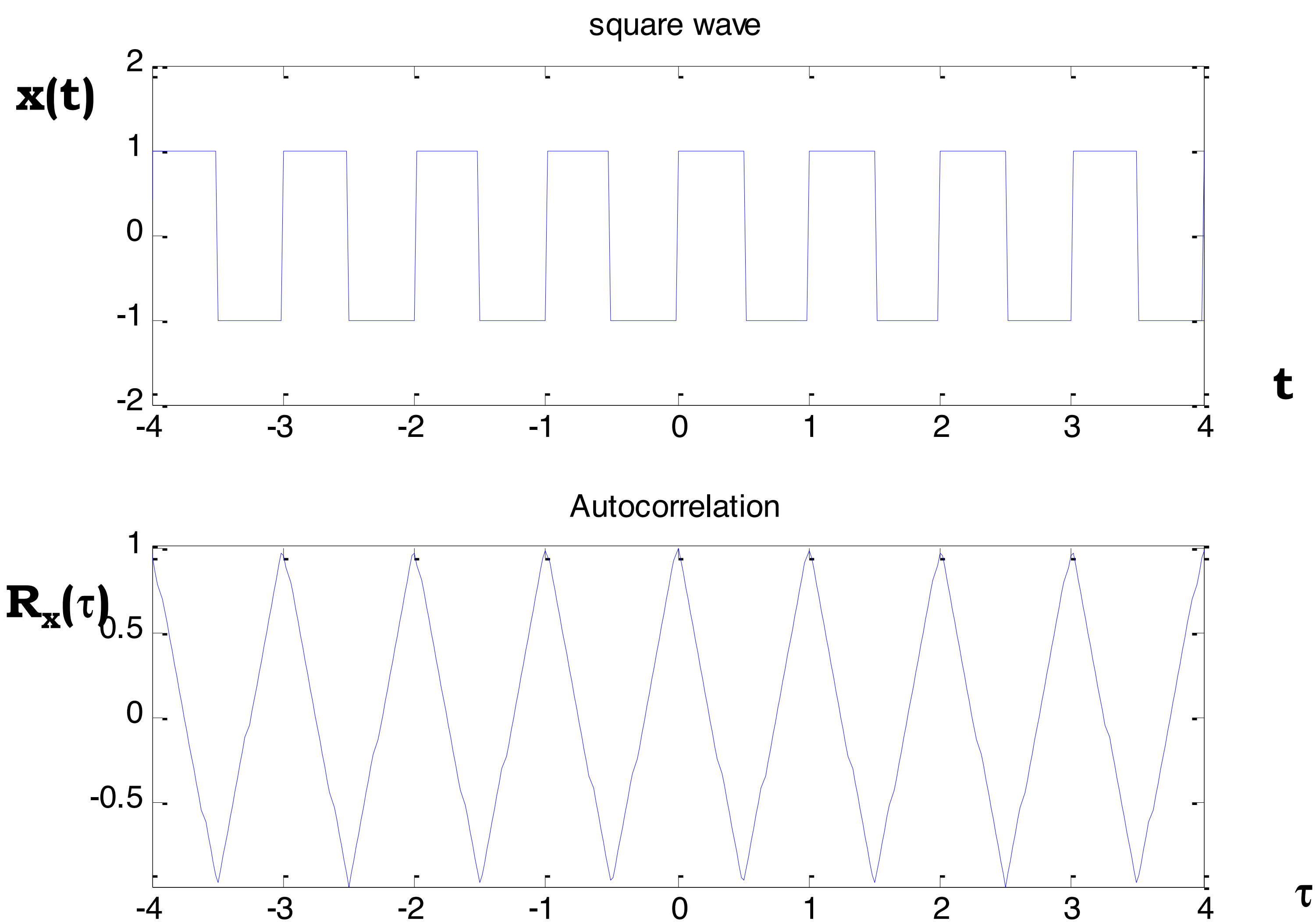
Correlation

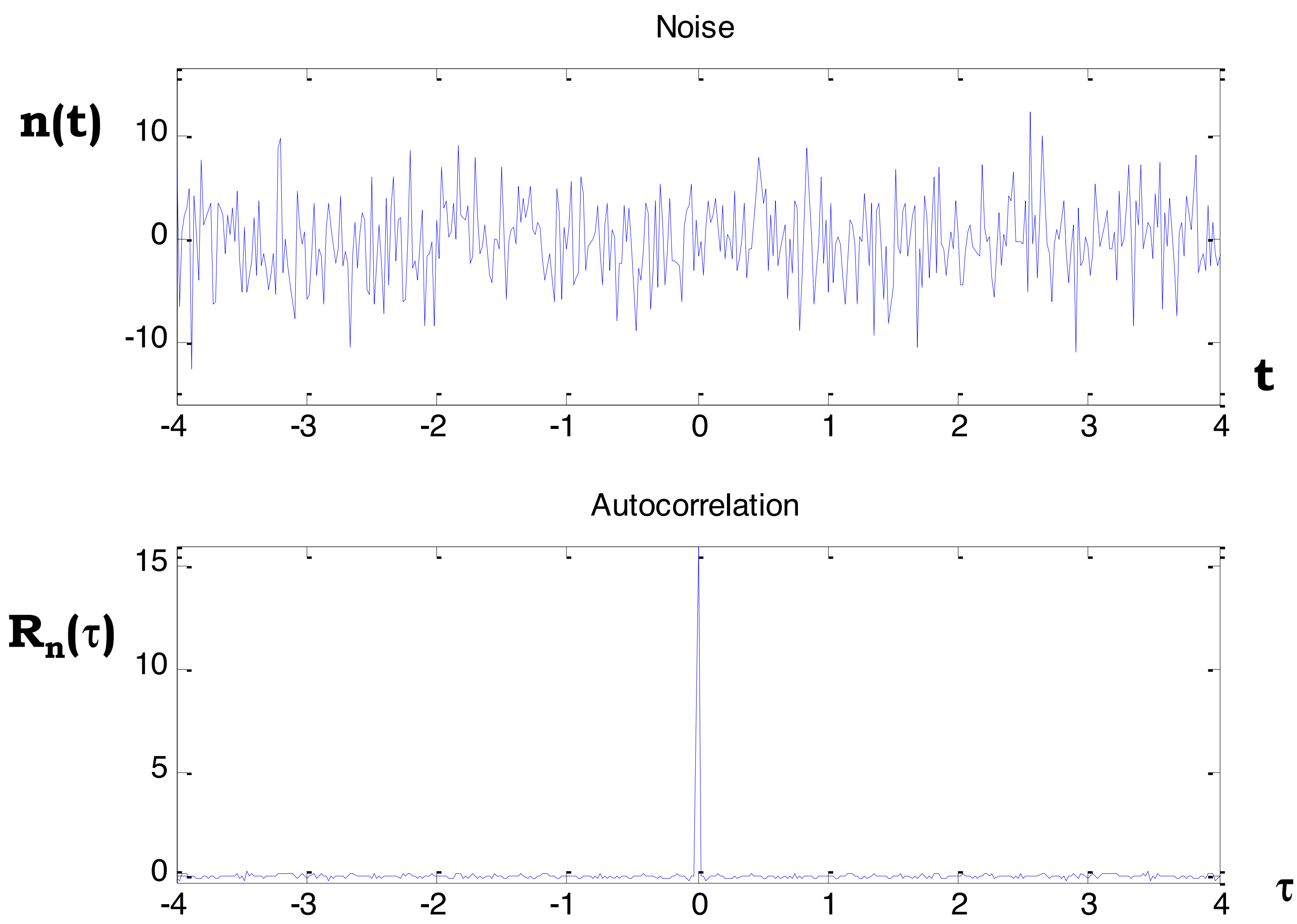
Exercise: Determine and sketch the autocorrelation function of a periodic square wave with peak to peak amplitude A, period T and mean value A/2.

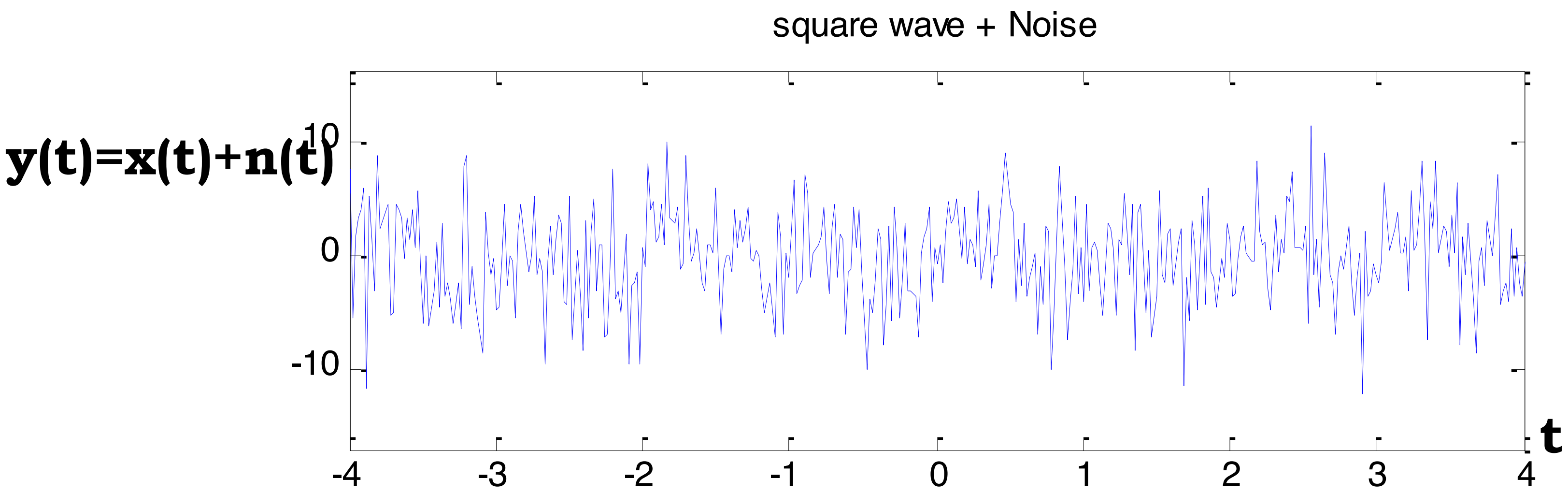


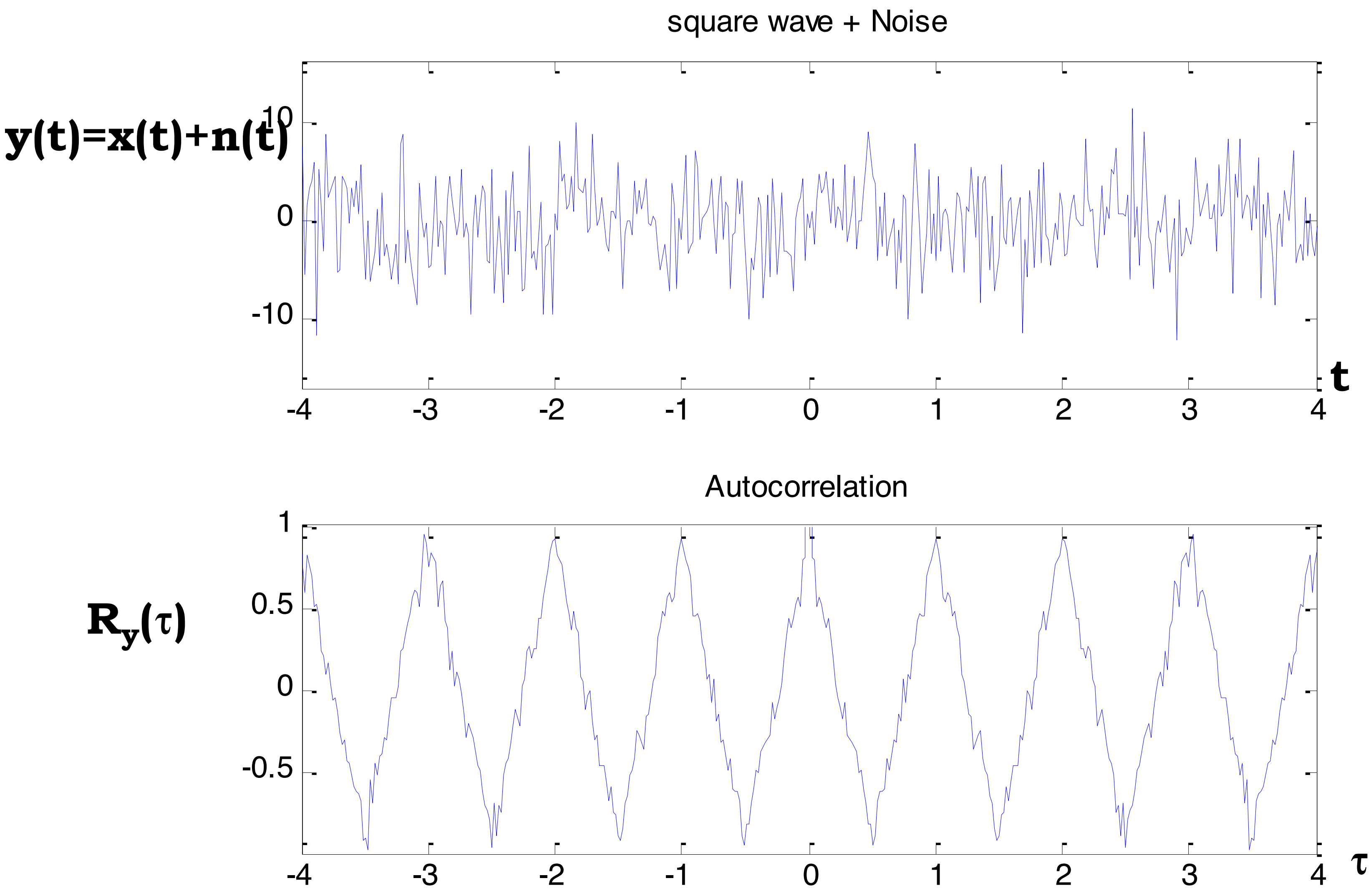



square wave
x(t)
2
1
0
-1
-2
-4
-3
-2
-1
0
1
2
3
4
t
Autocorrelation
$R_x(\tau)$
2
1.5
1
0.5
0
-4
-3
-2
-1
0
1
2
3
4
τ


square wave
x(t)
2
1
0
-1
-2
-4
-3
-2
-1
0
1
2
3
4
t
Autocorrelation
$R_x(\tau)$
1
0.5
0
-0.5
-4
-3
-2
-1
0
1
2
3
4
τ


Noise
n(t)
10
0
-10
t
-4
-3
-2
-1
0
1
2
3
4
Autocorrelation
$R_n(\tau)$
15
10
5
0
τ
-4
-3
-2
-1
0
1
2
3
4


square wave + Noise
y(t)=x(t)+n(t)
10
0
-10
-4
-3
-2
-1
0
1
2
3
4
t



square wave + Noise
y(t)=x(t)+n(t)
10
0
-10
t
-4
-3
-2
-1
0
1
2
3
4
Autocorrelation
$R_y(\tau)$
1
0.5
0
-0.5
τ
-4
-3
-2
-1
0
1
2
3
4

Autocorrelation applications

Q: What is autocorrelation used for?

- It widely used for signal analysis
- Used for detection an recognition of signals that are masked by additive noise.