EE 373, Spring 2009-2010

Communication Systems

Lectures 3&4 – Week 2

Probabilities, Random variables

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1.2 CLASSIFICATION OF SIGNALS

1.2.1 Deterministic and Random Signals

A signal can be classified as *deterministic,* meaning that there is no uncertainty with respect to its value at any time, or as *random,* meaning that there is some degree of uncertainty before the signal actually occurs. Deterministic signals or waveforms are modeled by explicit mathematical expressions, such as $x(t) = 5 \cos 10t$. For a random waveform it is *not* possible to write such an explicit expression. However, when examined over a long period, a random waveform, also referred to as a *random process,* may exhibit certain regularities that can be described in terms of probabilities and statistical averages. Such a model, in the form of a probabilistic description of the random process, is particularly useful for characterizing signals and noise in communication systems.

Classification of signals (1)

- (1) Deterministic and random signals
  - **Deterministic signal:** No uncertainty with respect to the signal value at any time.
  - **Random signal:** Some degree of uncertainty in signal values before it actually occurs.
    - Thermal noise in electronic circuits due to the random movement of electrons
    - Reflection of radio waves from different layers of ionosphere

1.2.3 Analog and Discrete Signals

An *analog signal* $x(t)$ is a continuous function of time; that is, $x(t)$ is uniquely defined for all t. An electrical analog signal arises when a physical waveform (e.g., speech) is converted into an electrical signal by means of a transducer. By comparison, a *discrete signal* $x(kT)$ is one that exists only at discrete times; it is characterized by a sequence of numbers defined for each time, kT, where k is an integer and T is a fixed time interval.

1.2.2 Periodic and Nonperiodic Signals

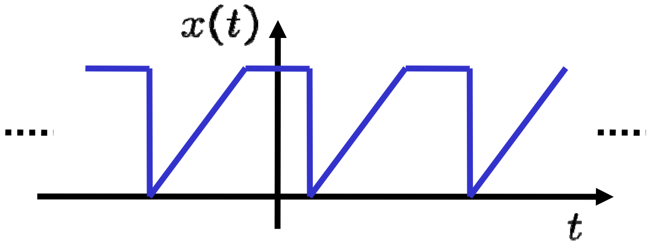
A signal $x(t)$ is called *periodic in time* if there exists a constant $T_0 > 0$ such that

$$x(t) = x(t + T_0) \qquad \text{for } -\infty < t < \infty \tag{1.2}$$

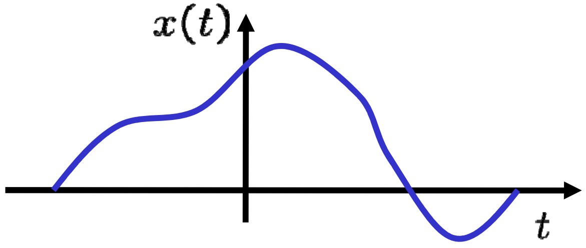
where t denotes time. The smallest value of T_0 that satisfies this condition is called the *period* of $x(t)$. The period T_0 defines the duration of one complete cycle of $x(t)$. A signal for which there is no value of T_0 that satisfies Equation (1.2) is called a *nonperiodic signal*.

Classification of signals (2,3)…

- (2) Periodic and non-periodic signals

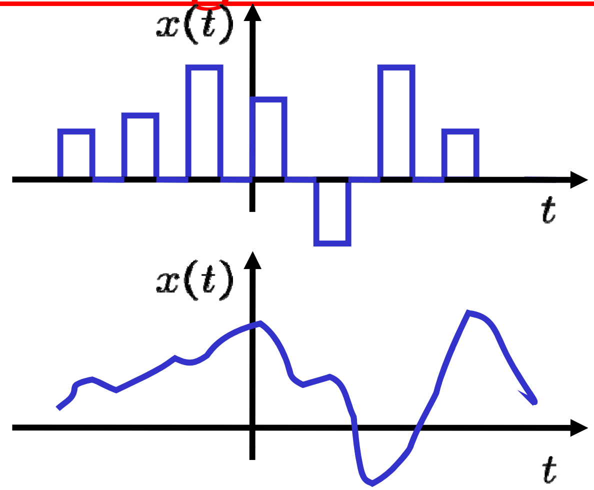


A periodic signal

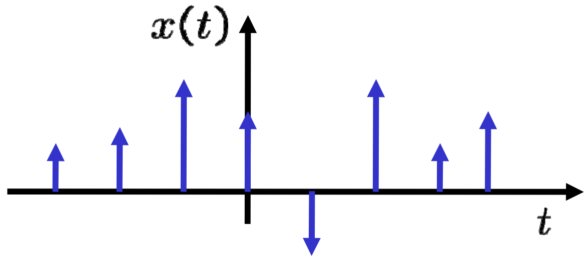


A non-periodic signal

- (3) Analog and discrete signals



Analog signals



A discrete signal

1.2.4 Energy and Power Signals

An electrical signal can be represented as a voltage $v(t)$ or a current $i(t)$ with instantaneous power $p(t)$ across a resistor $\mathcal{R}$ defined by

$$p(t) = \frac{v^2(t)}{\mathcal{R}} \tag{1.3a}$$

or

$$p(t) = i^2(t)\mathcal{R} \tag{1.3b}$$

In communication systems, power is often normalized by assuming $\mathcal{R}$ to be 1 Ω, although $\mathcal{R}$ may be another value in the actual circuit. If the actual value of the power is needed, it is obtained by "denormalization" of the normalized value. For the normalized case, Equations 1.3a and 1.3b have the same form. Therefore, regardless of whether the signal is a voltage or current waveform, the normalization convention allows us to express the instantaneous power as

$$p(t) = x^2(t) \tag{1.4}$$

where $x(t)$ is either a voltage or a current signal. The energy dissipated during the time interval $(-T/2, T/2)$ by a real signal with instantaneous power expressed by Equation (1.4) can then be written as

$$E_x^T = \int_{-T/2}^{T/2} x^2(t)\, dt \tag{1.5}$$

and the average power dissipated by the signal during the interval is

$$P_x^T = \frac{1}{T} E_x^T = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \tag{1.6}$$

In analyzing communication signals, it is often desirable to deal with the *waveform energy.* We classify $x(t)$ as an *energy signal* if, and only if, it has nonzero but finite energy ($0 < E_x < \infty$) for all time, where

$$E_x = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt = \int_{-\infty}^{\infty} x^2(t)\, dt \tag{1.7}$$

In the real world, we always transmit signals having finite energy ($0 < E_x < \infty$). However, in order to describe *periodic signals,* which by definition [Equation (1.2)] exist for all time and thus have infinite energy, and in order to deal with random signals that have infinite energy, it is convenient to define a class of signals called *power signals.* A signal is defined as a power signal if, and only if, it has finite but nonzero power ($0 < P_x < \infty$) for all time, where

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \tag{1.8}$$

The energy and power classifications are mutually exclusive. An energy signal has finite energy but *zero average power*, whereas a power signal has finite average power but *infinite energy*. A waveform in a system may be constrained in either its power or energy values. As a general rule, periodic signals and random signals are classified as power signals, while signals that are both deterministic and nonperiodic are classified as energy signals [1, 2].

Classification of signals (4)..

- (4) Energy and power signals
  - A signal is an energy signal if, and only if, it has **nonzero but finite energy for all time:**

$$E_x = \lim_{T \to \infty} \int_{T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\boxed{(0 < E_x < \infty)}$$

  - A signal is a power signal if, and only if, it has **finite but nonzero power for all time**:

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{(0 < P_x < \infty)}$$

General rule:

- **Periodic and random signals** = **power signals.**
- **deterministic and non-periodic** = **energy signals**.

1.2.5 The Unit Impulse Function

A useful function in communication theory is the unit impulse or *Dirac delta function* $\delta(t)$. The impulse function is an abstraction—an infinitely large amplitude pulse, with zero pulse width, and unity weight (area under the pulse), concentrated at the point where its argument is zero. The unit impulse is characterized by the following relationships:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.9}$$

$$\delta(t) = 0 \qquad \text{for } t \neq 0 \tag{1.10}$$

$$\delta(t) \text{ is unbounded at } t = 0 \tag{1.11}$$

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0) \tag{1.12}$$

Equation (1.12) is known as the *sifting* or *sampling property* of the unit impulse function; the unit impulse multiplier selects a sample of the function $x(t)$ evaluated at $t = t_0$.

1.3 SPECTRAL DENSITY

The *spectral density* of a signal characterizes the distribution of the signal's energy or power in the frequency domain. This concept is particularly important when considering filtering in communication systems. We need to be able to evaluate the signal and noise at the filter output. The energy spectral density (ESD) or the power spectral density (PSD) is used in the evaluation.

1.3.1 Energy Spectral Density

The total energy of a real-valued energy signal $x(t)$, defined over the interval, $(-\infty, \infty)$, is described by Equation (1.7). Using Parseval's theorem [1], we can relate the energy of such a signal expressed in the time domain to the energy expressed in the frequency domain, as

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df \tag{1.13}$$

where $X(f)$ is the Fourier transform of the nonperiodic signal $x(t)$. (For a review of Fourier techniques, see Appendix A.) Let $\psi_x(f)$ denote the squared magnitude spectrum, defined as

$$\psi_x(f) = |X(f)|^2 \tag{1.14}$$

The quantify $\psi_x(f)$ is the waveform *energy spectral density* (ESD) of the signal $x(t)$. Therefore, from Equation (1.13), we can express the total energy of $x(t)$ by integrating the spectral density with respect to frequency:

$$E_x = \int_{-\infty}^{\infty} \psi_x(f)\,df \tag{1.15}$$

1.3.2 Power Spectral Density

The average power P_x of a real-valued power signal $x(t)$ is defined in Equation (1.8). If $x(t)$ is a *periodic signal* with period T_0, it is classified as a power signal. The expression for the average power of a periodic signal takes the form of Equation (1.6), where the time average is taken over the signal period T_0, as follows:

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt \tag{1.17a}$$

Parseval's theorem for a real-valued periodic signal [1] takes the form

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{1.17b}$$

where the $|c_n|$ terms are the complex Fourier series coefficients of the periodic signal. (See Appendix A.)

signal. (See Appendix A.)

To apply Equation (1.17b), we need only know the magnitude of the coefficients, $|c_n|$. The *power spectral density* (PSD) function $G_x(f)$ of the periodic signal $x(t)$ is a real, even, and nonnegative function of frequency that gives the distribution of the power of $x(t)$ in the frequency domain, defined as

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0) \tag{1.18}$$

Equation (1.18) defines the power spectral density of a periodic signal $x(t)$ as a succession of the weighted delta functions. Therefore, the PSD of a periodic signal is a discrete function of frequency. Using the PSD defined in Equation (1.18), we can now write the average normalized power of a real-valued signal as

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = 2 \int_{0}^{\infty} G_x(f)\, df \tag{1.19}$$

Equation (1.18) describes the PSD of periodic (power) signals only. If $x(t)$ is a nonperiodic signal it *cannot* be expressed by a Fourier series, and if it is a nonperiodic power signal (having infinite energy) it *may not* have a Fourier transform. However, we may still express the power spectral density of such signals in the *limiting sense*. If we form a *truncated version* $x_T(t)$ of the nonperiodic power signal $x(t)$ by observing it only in the interval $(-T/2, T/2)$, then $x_T(t)$ has finite energy and has a proper Fourier transform $X_T(f)$. It can be shown [2] that the power spectral density of the nonperiodic $x(t)$ can then be defined in the limit as

$$G_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2 \tag{1.20}$$

Example 1.1 Average Normalized Power

(a) Find the average normalized power in the waveform, $x(t) = A \cos 2\pi f_0 t$, using time averaging.

(b) Repeat part (a) using the summation of spectral coefficients.

Solution

(a) Using Equation (1.17a), we have

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$$

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A^2 \cos^2 2\pi f_0 t \; dt$$

$$= \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} (1 + \cos 4\pi f_0 t)\, dt$$

$$= \frac{A^2}{2T_0} (T_0) = \frac{A^2}{2}$$

(b) Using Equations (1.18) and (1.19) gives us

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = 2 \int_{0}^{\infty} G_x(f)\, df$$

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0)$$

$$\left.\begin{aligned} c_1 &= c_{-1} = \frac{A}{2} \\ c_n &= 0 \qquad \text{for } n = 0, \pm 2, \pm 3, \ldots \end{aligned}\right\} \quad \text{(see Appendix A)}$$

$$G_x(f) = \left(\frac{A}{2}\right)^2 \delta(f - f_0) + \left(\frac{A}{2}\right)^2 \delta(f + f_0)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = \frac{A^2}{2}$$

1.4 AUTOCORRELATION

1.4.1 Autocorrelation of an Energy Signal

Correlation is a matching process; *autocorrelation* refers to the matching of a signal with a delayed version of itself. The autocorrelation function of a real-valued energy signal $x(t)$ is defined as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt \qquad \text{for } -\infty < \tau < \infty \tag{1.21}$$

The autocorrelation function $R_x(\tau)$ provides a measure of how closely the signal matches a copy of itself as the copy is shifted τ units in time. The variable τ plays the role of a scanning or searching parameter. $R_x(\tau)$ is not a function of time; it is only a function of the time difference τ between the waveform and its shifted copy.

The autocorrelative function of a real-valued *energy* signal has the following properties:

1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $R_x(\tau) \leq R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow \psi_x(f)$	autocorrelation and ESD form a Fourier transform pair, as designated by the double-headed arrows
4. $R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt$	value at the origin is equal to the energy of the signal

If items 1 through 3 are satisfied, $R_x(\tau)$ satisfies the properties of an autocorrelation function. Property 4 can be derived from property 3 and thus need not be included as a basic test.

1.4.2 Autocorrelation of a Periodic (Power) Signal

The autocorrelation function of a real-valued power signal $x(t)$ is defined as

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.22}$$

When the power signal $x(t)$ is periodic with period T_0, the time average in Equation (1.22) may be taken over a *single period* T_0, and the autocorrelation function can be expressed as

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.23}$$

The autocorrelation function of a real-valued *periodic* signal has properties similar to those of an energy signal:

1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $R_x(\tau) \le R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow G_x(f)$	autocorrelation and PSD form a Fourier transform pair
4. $R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$	value at the origin is equal to the average power of the signal

Spectral density

- Energy signals:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad X(f) = \mathcal{F}[x(t)]$$

  - Energy spectral density (ESD):

$$\Psi_x(f) = |X(f)|^2$$

- Power signals:

$$P_x = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \{c_n\} = \mathcal{F}[x(t)]$$

  - Power spectral density (PSD):

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_0) \qquad f_0 = 1/T_0$$

- Random process:
  - Power spectral density (PSD):

$$G_X(f) = \mathcal{F}[R_X(\tau)]$$

Autocorrelation

- Autocorrelation of an energy signal

$$R_x(\tau) = x(\tau) \star x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a power signal

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

  – For a periodic signal:

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

1.5 RANDOM SIGNALS

1.5.1 Random Variables

Let a *random variable* $X(A)$ represent the functional relationship between a random event A and a real number. For notational convenience, we shall designate the random variable by X, and let the functional dependence upon A be implicit. The random variable may be discrete or continuous. The *distribution function* $F_X(x)$ of the random variable X is given by

$$F_X(x) = P(X \leq x) \tag{1.24}$$

where $P(X \leq x)$ is the probability that the value taken by the random variable X is less than or equal to a real number x. The distribution function $F_X(x)$ has the following properties:

1. $0 \leq F_X(x) \leq 1$

2. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

3. $F_X(-\infty) = 0$

4. $F_X(+\infty) = 1$

Another useful function relating to the random variable X is the *probability density function* (pdf), denoted

$$p_X(x) = \frac{dF_X(x)}{dx} \tag{1.25a}$$

As in the case of the distribution function, the pdf is a function of a real number x. The name "density function" arises from the fact that the probability of the event $x_1 \leq X \leq x_2$ equals

$$\begin{aligned} P(x_1 \leq X \leq x_2) &= P(X \leq x_2) - P(X \leq x_1) \\ &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} p_X(x)\, dx \end{aligned} \tag{1.25b}$$

From Equation (1.25b), the probability that a random variable X has a value in some very narrow range between x and $x + \Delta x$ can be approximated as

$$P(x \le X \le x + \Delta x) \approx p_X(x)\Delta x \tag{1.25c}$$

Thus, in the limit as Δx approaches zero, we can write

$$P(X = x) = p_X(x)dx \tag{1.25d}$$

The probability density function has the following properties:

1. $p_X(x) \ge 0$.
2. $\int_{-\infty}^{\infty} p_X(x)\,dx = F_X(+\infty) - F_X(-\infty) = 1$.

Thus, a probability density function is always a nonnegative function with a total area of one. Throughout the book we use the designation $p_X(x)$ for the probability density function of a *continuous* random variable. For ease of notation, we will often omit the subscript X and write simply $p(x)$. We will use the designation $p(X = x_i)$ for the probability of a random variable X, where X can take on *discrete* values only.

1.5.1.1 Ensemble Averages

The *mean value* m_X, or *expected value* of a random variable X, is defined by

$$m_X = \mathbf{E}\{X\} = \int_{-\infty}^{\infty} x p_X(x)\, dx \qquad (1.26)$$

where $\mathbf{E}\{\cdot\}$ is called the *expected value operator.* The *nth moment* of a probability distribution of a random variable X is defined by

$$\mathbf{E}\{X^n\} = \int_{-\infty}^{\infty} x^n p_X(x)\, dx \qquad (1.27)$$

For the purposes of communication system analysis, the most important moments of X are the first two moments. Thus, $n = 1$ in Equation (1.27) gives m_X as discussed above, whereas $n = 2$ gives the mean-square value of X, as follows:

$$\mathbf{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \qquad (1.28)$$

We can also define *central moments,* which are the moments of the difference between X and m_X. The second central moment, called the *variance* of X, is defined as

$$\operatorname{var}(X) = \mathbf{E}\{X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x)\, dx \tag{1.29}$$

The variance of X is also denoted as σ_X^2, and its square root, σ_X, is called the *standard deviation* of X. Variance is a measure of the "randomness" of the random variable X. By specifying the variance of a random variable, we are constraining the width of its probability density function. The variance and the mean-square value are related by

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}\{X^2 - 2m_X X + m_X^2\} \\ &= \mathbf{E}\{X^2\} - 2m_X \mathbf{E}\{X\} + m_X^2 \\ &= \mathbf{E}\{X^2\} - m_X^2\end{aligned}$$

Thus, the variance is equal to the difference between the mean-square value and the square of the mean.

Revision of Probability theory and random variables

Why is probability important?

- Random Variables and Processes let us talk about quantities and signals which are unknown in advance:
- The data sent through a communication system is modeled as random
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability.

Sample Space and Probability

- *Random experiment*: its outcome, for some reason, cannot be predicted with certainty.
- Examples: throwing a die, flipping a coin and drawing a card from a deck.
- *Sample space*: the set of all possible outcomes, denoted by Ω. Outcomes are denoted by ω's and each ω lies in Ω, i.e., $\omega \in \Omega$.
- A sample space can be *discrete* or *continuous*.
- *Events* are subsets of the sample space for which measures of their occurrences, called probabilities, can be defined or determined.

Random Events

- When we conduct a random experiment, we can use set notation to describe possible outcomes.
- Example: Roll a six-sided die.

  Possible Outcomes: $S = \{1,2,3,4,5,6\}$
- An event is any subset of possible outcomes: $A = \{1,2\}$
- The complementary event: $\overline{A} = S - A = \{3,4,5,6\}$
- The set of all outcomes is the certain event: S
- The null event: ϕ
- Transmitting a data bit is also an experiment

Probability

- The probability *P(A)* is a number which measures the likelihood of the event A.

Axioms of Probability:

- No event has probability less than zero: $P(A) \geq 0$
- $P(A) \leq 1$ and $P(A) = 1 \Leftrightarrow A = S$
- Let A and B be two events such that: $A \cap B = \phi$

  Then: $P(A \cup B) = P(A) + P(B)$
- All other laws of probability follow from these axioms

Relationships Between Random Events

- Joint Probability: $P(A,B) = P(A \cap B)$
  - Probability that both A and B occur
- Conditional Probability: $P(A|B) = \dfrac{P(A,B)}{P(B)}$
  - Probability that A will occur given that B <u>has</u> occurred
- Statistical Independence:
  - Events A and B are statistically independent if:

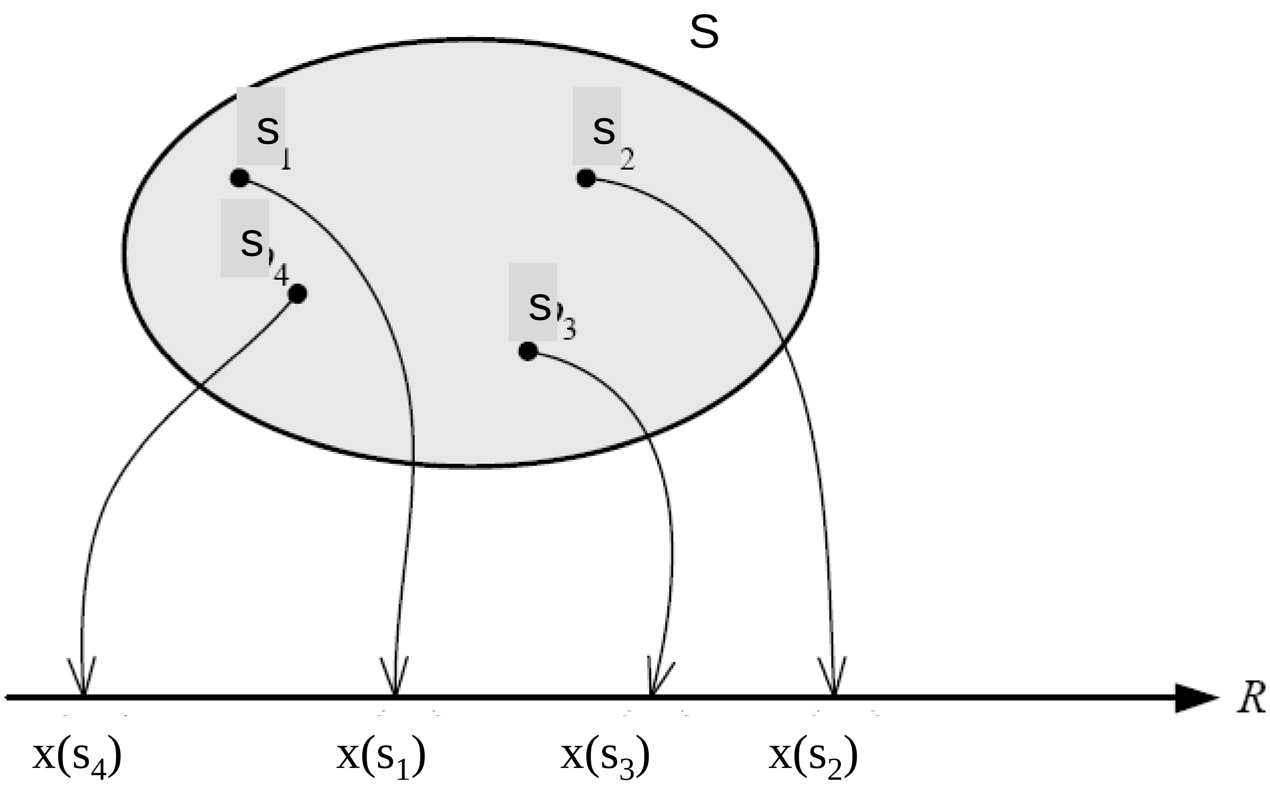
    $$P(A,B) = P(A) \cdot P(B)$$
  - If A and B are independent then:

    $$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

Random Variables

- A random variable *X(s)* is a real-valued function of the underlying event space: $s \in S$
- A random variable may be:
  – Discrete-valued: range is finite (e.g. $\{0,1\}$) or countably infinite (e.g., $\{1,2,3,\ldots\}$)
  – Continuous-valued - range is uncountably infinite (e.g. $\Re$)
- A random variable may be described by:
  – A name: X
  – It's range: $X \in \Re$
  – A description of its distribution

Random Variables



- A random variable is a *mapping* from the sample space S the set of real numbers.
- We shall denote random variables by Capital , i.e. X, Y e while individual or specific values of the mapping X are denoted by x(s).

Probability Distribution Function (PDF)

- Also called Cumulative Distribution Function (CDF)
- Definition: $F_X(x) = F(x) = P(X \leq x)$
- Properties:

  $F(x)$ is monotonically nondecreasing

  $F(-\infty) = 0$

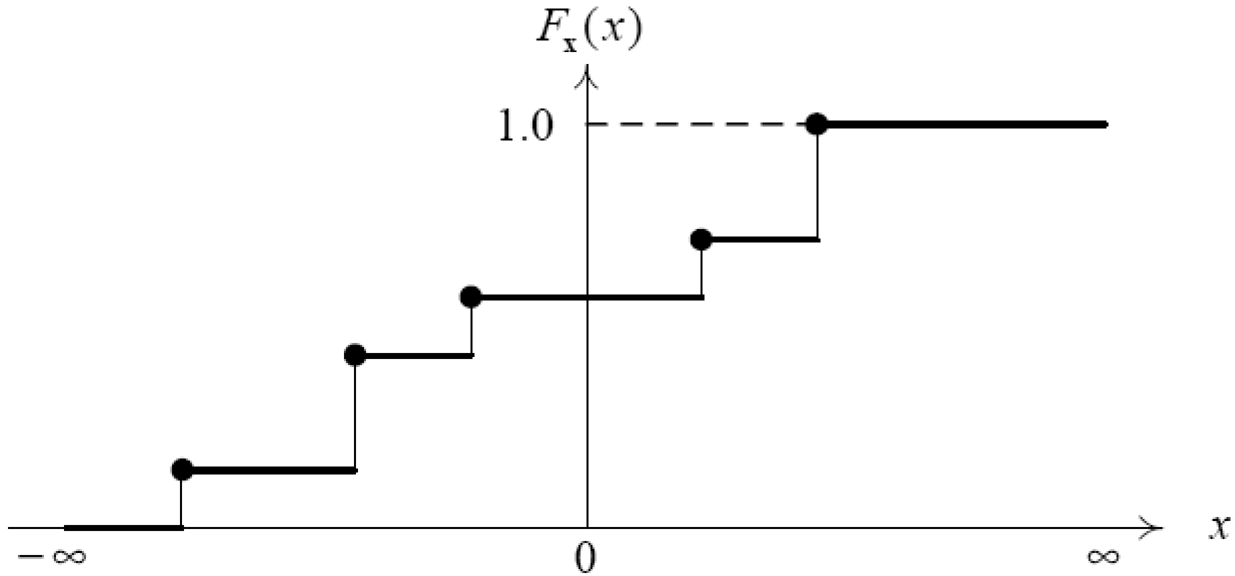
  $F(\infty) = 1$

  $P(a < X \leq b) = F(b) - F(a)$
- While the PDF completely defines the distribution of a random variable, we will usually work with the pdf or pmf

Typical Plots of cdf I

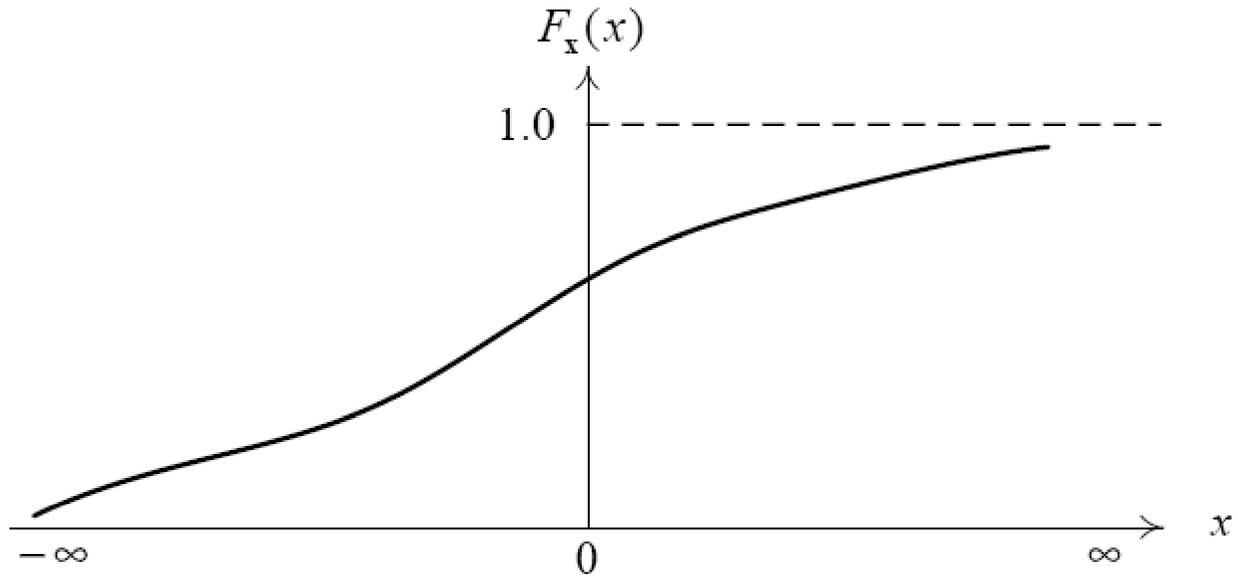
A random variable can be *discrete*, *continuous* or *mixed*.

(a)

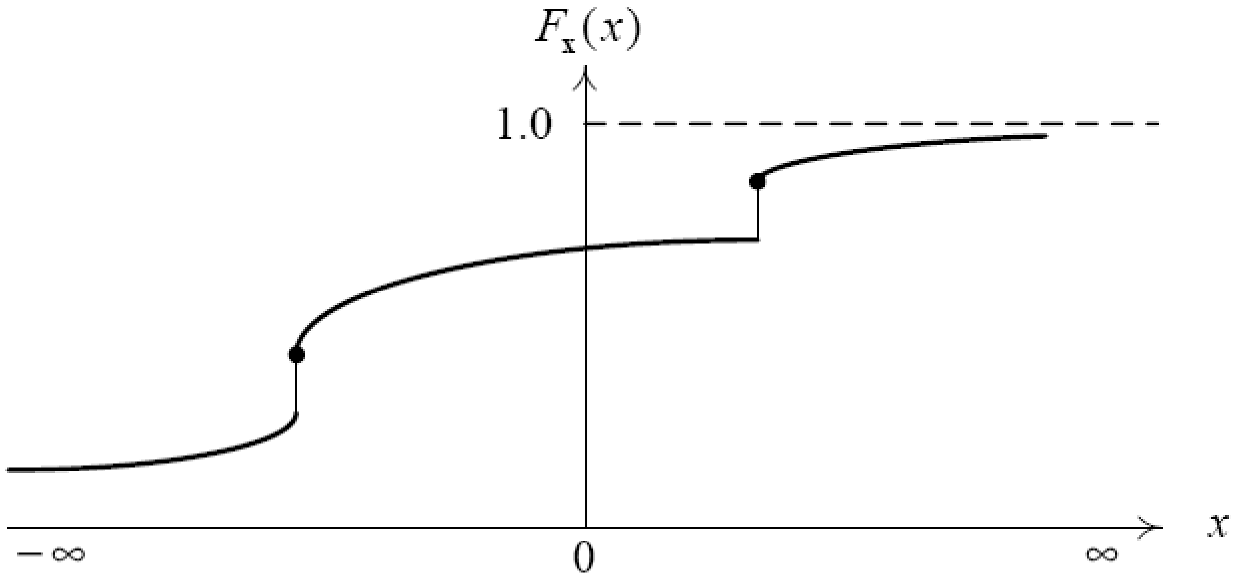


Typical Plots of cdf II

(b)



(c)



Probability Density Function (pdf)

- The pdf is defined as the derivative of the cdf:

$$p(x) = \frac{\mathrm{d}F_{\mathbf{x}}(x)}{\mathrm{d}x}.$$

- It follows that:

$$P(x_1 \leq \mathbf{x} \leq x_2) = P(\mathbf{x} \leq x_2) - P(\mathbf{x} \leq x_1)$$
$$= F_{\mathbf{x}}(x_2) - F_{\mathbf{x}}(x_1) = \int_{x_1}^{x_2} p(x)\, \mathrm{d}x.$$

- Basic properties of pdf:
  1. $p(x) \geq 0$.
  2. $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$.
  3. In general, $P(\mathbf{x} \in \mathcal{A}) = \int_{\mathcal{A}} p(x)\mathrm{d}x$.
- For discrete random variables, it is more common to defin[e] the *probability mass function* (pmf): $p_i = P(\mathbf{x} = x_i)$.
- Note that, for all i, one has $p_i \geq 0$ and $\sum p_i = 1$.

Probability Density Function (pdf)

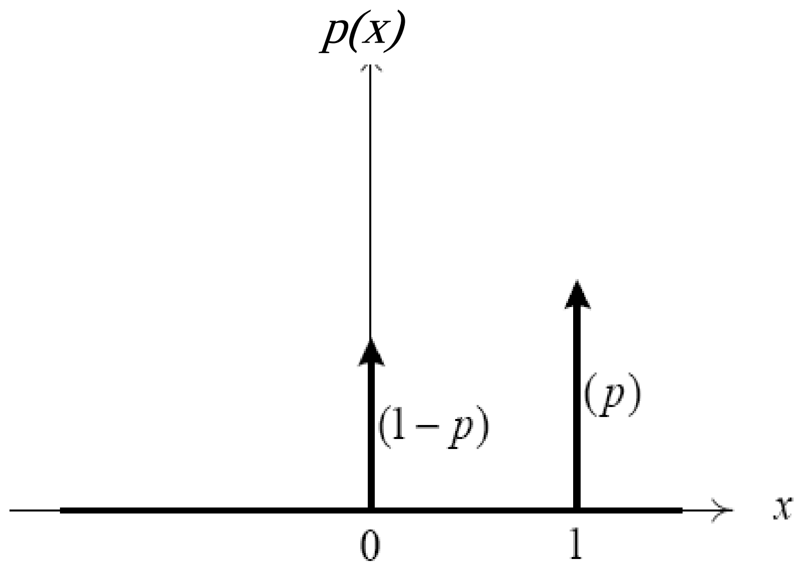
- Defn: $p_X(x) = \dfrac{dF_X(x)}{dx}$ or $p(x) = \dfrac{dF(x)}{dx}$
- Interpretations:
  - pdf measures how fast PDF is increasing or how likely a random variable is to lie at a particular value
- Properties:

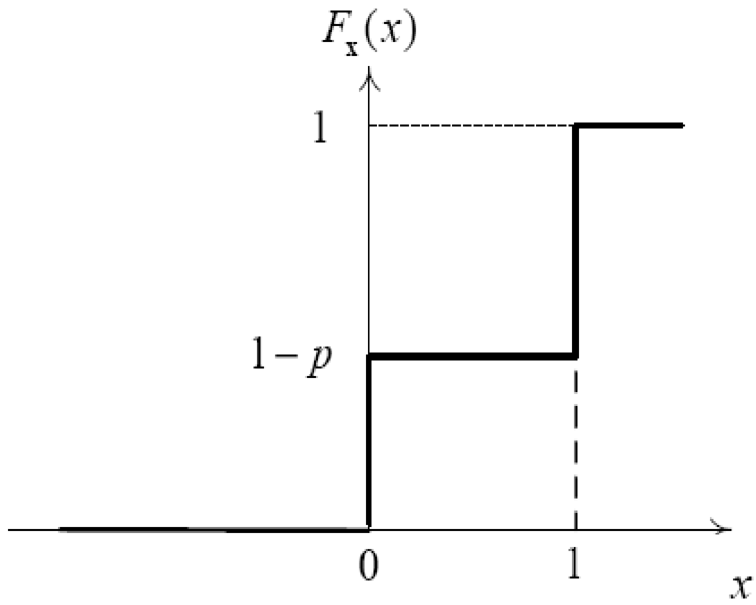
$$p(x) \geq 0$$

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

$$P(a < X \leq b) = \int_{a}^{b} p(x)dx$$

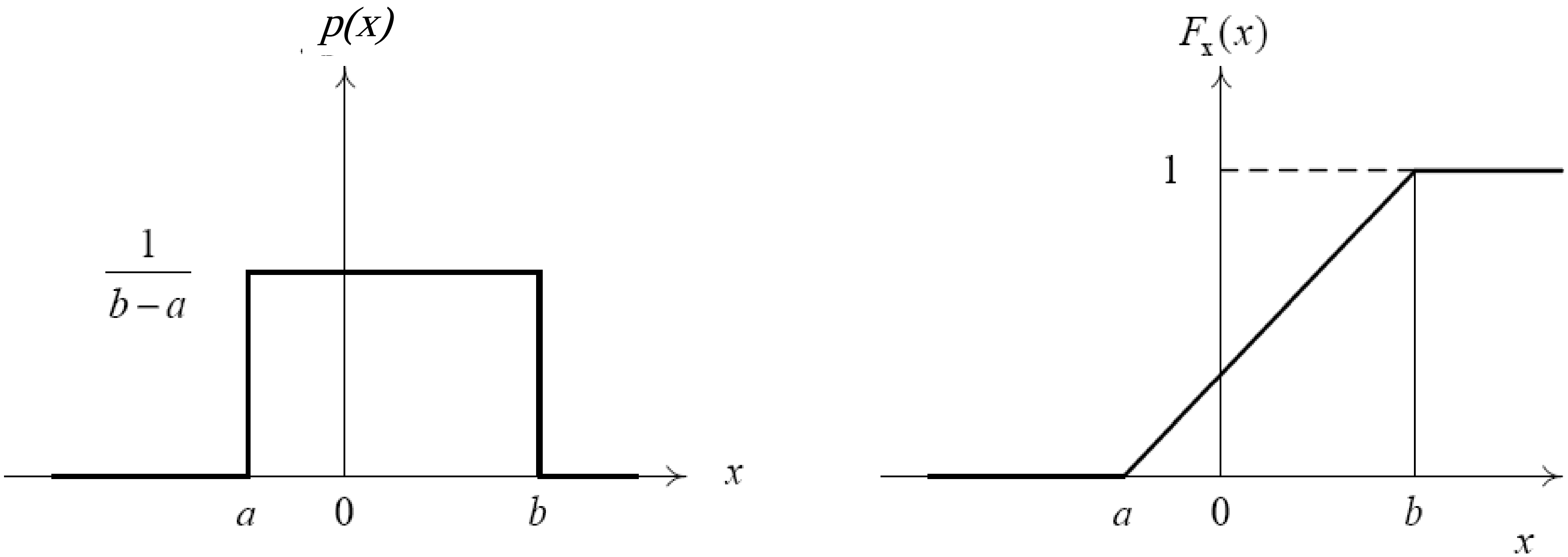
Bernoulli Random Variable





- A discrete random variable that takes two values 1 and 0 with probabilities p and $1 - p$.
- Good model for a binary data source whose output is 1 or 0.
- Can also be used to model the channel errors.

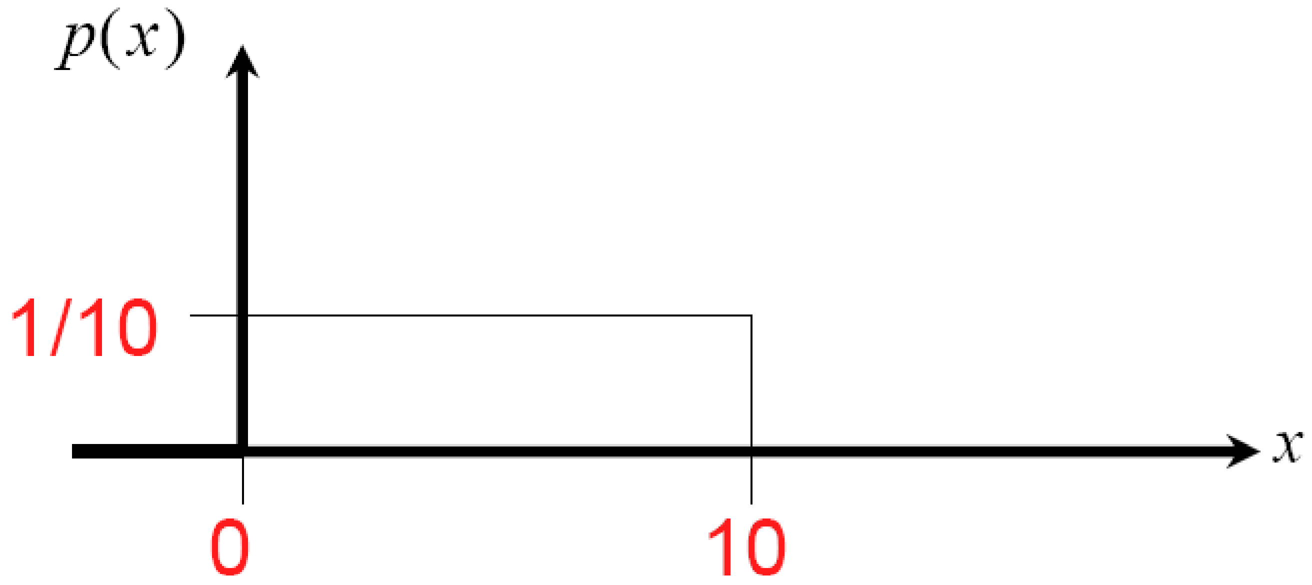
Uniform Random Variable



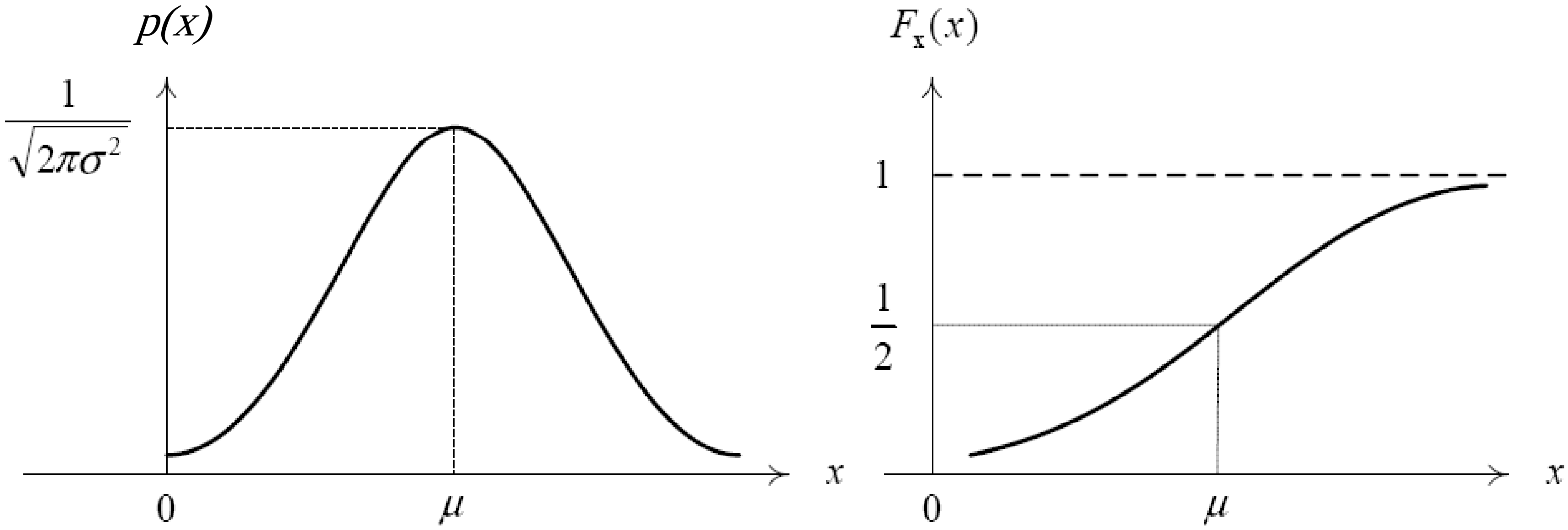
- A continuous random variable that takes values between a and b with equal probabilities over intervals of equal length.
- The phase of a received sinusoidal carrier is usually modeled as a uniform random variable between 0 and 2π. Quantization error is also typically modeled as uniform.

Example #1: Uniform pdf

- $$p(x) = \begin{cases} 1/10, & 0 \le x \le 10 \\ 0, & \text{else} \end{cases}$$



Gaussian (or Normal) Random Variable



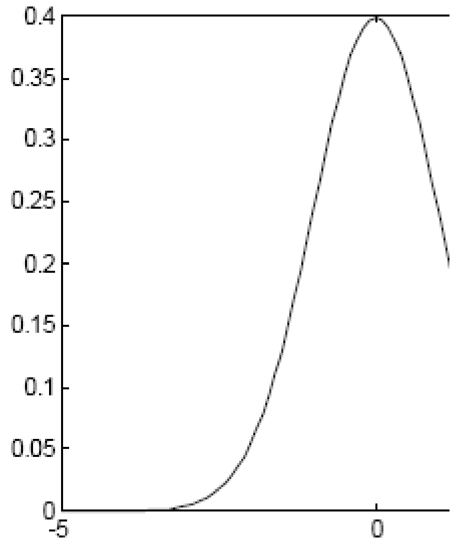
- A continuous random variable whose pdf is:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \qquad (10)$$

  μ and σ^2 are parameters. Usually denoted as $\mathcal{N}(\mu, \sigma^2)$.
- Most important and frequently encountered random variable in communications.

Example #2: Gaussian pdf

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}$$



- A Gaussian random variable is completely determined by its mean and variance

Example #3 - Rayleigh pdf

- Let:

$$R = \sqrt{X_1^2 + X_2^2}$$

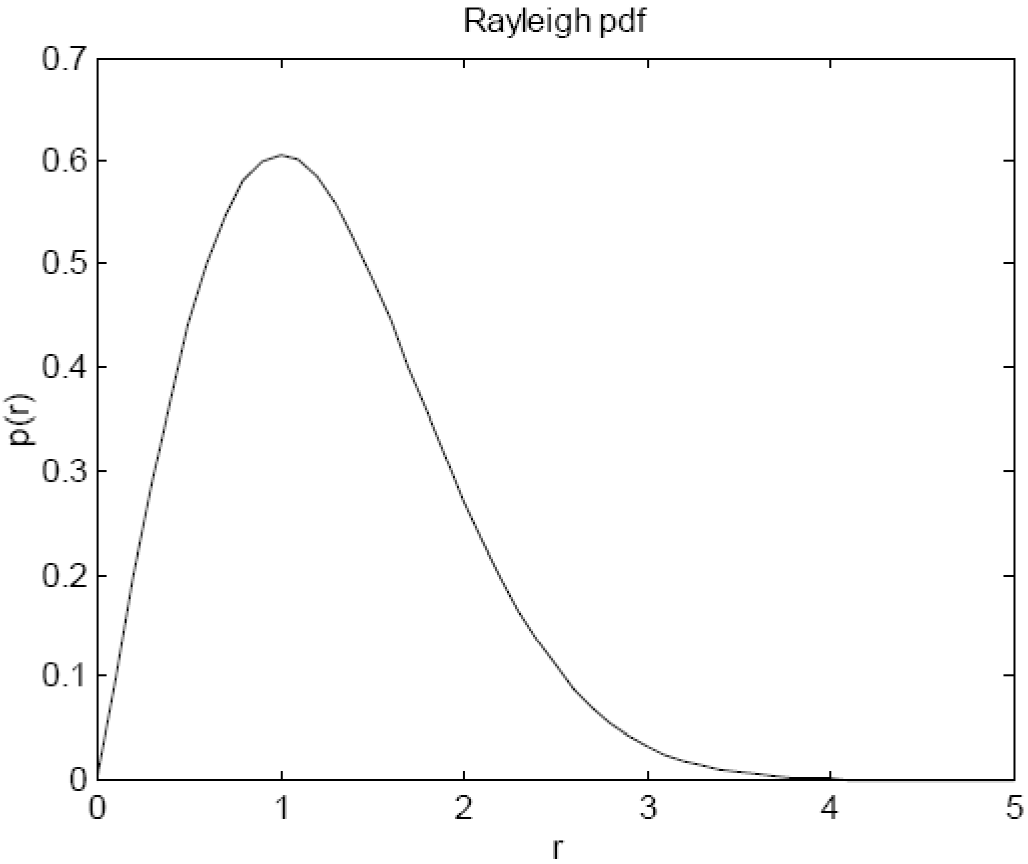
  where X_1 and X_2 are Gaussian with mean 0 and variance σ^2

- Then R is a Rayleigh random variable with pdf:

$$p_R(r) = \frac{r}{\sigma^2} e^{-r^2 / 2\sigma^2}$$

- Rayleigh pdf's are frequently used to model fading when no line of site signal is present

Rayleigh pdf



Expected Values

- Expected values are a shorthand way of describing a random variable
- The most important examples are:
  - Mean: $E(X) = m_X = \int_{-\infty}^{\infty} x p(x) dx$
  - Variance: $E\left(\left[X - m_X\right]^2\right) = \int_{-\infty}^{\infty} (x - m_X)^2 p(x) dx$
- The expectation operator works with any function:

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p(x) dx$$

Expectation of Random Variables I

Statistical averages, or *moments*, play an important role in the characterization of the random variable.

- The *expected value* (also called the mean value, first moment) of the random variable $\mathbf{x}$ is defined as

$$m_{\mathbf{x}} = E\{\mathbf{x}\} \equiv \int_{-\infty}^{\infty} x \; p(x) \; \mathrm{d}x, \tag{13}$$

  where E denotes the *statistical expectation operator*.

- In general, the nth moment of $\mathbf{x}$ is defined as

$$E\{\mathbf{x}^n\} \equiv \int_{-\infty}^{\infty} x^n \; p(x) \; \mathrm{d}x. \tag{14}$$

- For $n = 2$, $E\{\mathbf{x}^2\}$ is known as the mean-squared value of the random variable.

Expectation of Random Variables II

- The nth *central moment* of the random variable $\mathbf{x}$ is:

$$E\{\mathbf{y}\} = E\{(\mathbf{x} - m_{\mathbf{x}})^n\} = \int_{-\infty}^{\infty} (x - m_{\mathbf{x}})^n \, p(x) \, \mathrm{d}x. \quad (15)$$

- When $n = 2$ the central moment is called the *variance*, commonly denoted as $\sigma_{\mathbf{x}}^2$:

$$\sigma_{\mathbf{x}}^2 = \mathrm{var}(\mathbf{x}) = E\{(\mathbf{x} - m_{\mathbf{x}})^2\} = \int_{-\infty}^{\infty} (x - m_{\mathbf{x}})^2 \, p(x) \, \mathrm{d}x. \quad (16)$$

- The variance provides a measure of the variable's "randomness".
- The mean and variance of a random variable give a *partial description* of its pdf.

Expectation of Random Variables III

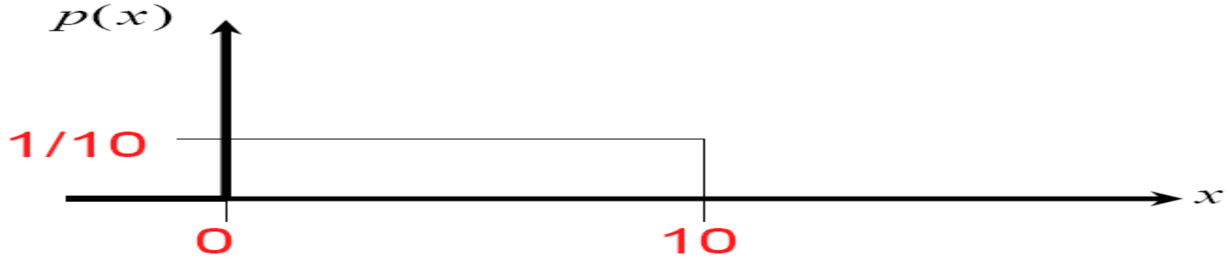
- Relationship between the variance, the first and second moments:

$$\sigma_{\mathbf{x}}^2 = E\{\mathbf{x}^2\} - [E\{\mathbf{x}\}]^2 = E\{\mathbf{x}^2\} - m_{\mathbf{x}}^2. \qquad (17)$$

- An electrical engineering interpretation: *The AC power equals total power minus DC power.*
- The square-root of the variance is known as the *standard deviation*, and can be interpreted as the root-mean-squared (RMS) value of the AC component.

Example #1: Uniform pdf

- $$p(x) = \begin{cases} 1/10, & 0 \le x \le 10 \\ 0, & \text{else} \end{cases}$$



Example #1 (continued)

- Mean:

$$m_x = \int_{-\infty}^{\infty} x \cdot p(x)dx = \int_0^{10} x \cdot \frac{1}{10}dx = \left[\frac{x^2}{20}\right]_0^{10} = 5$$

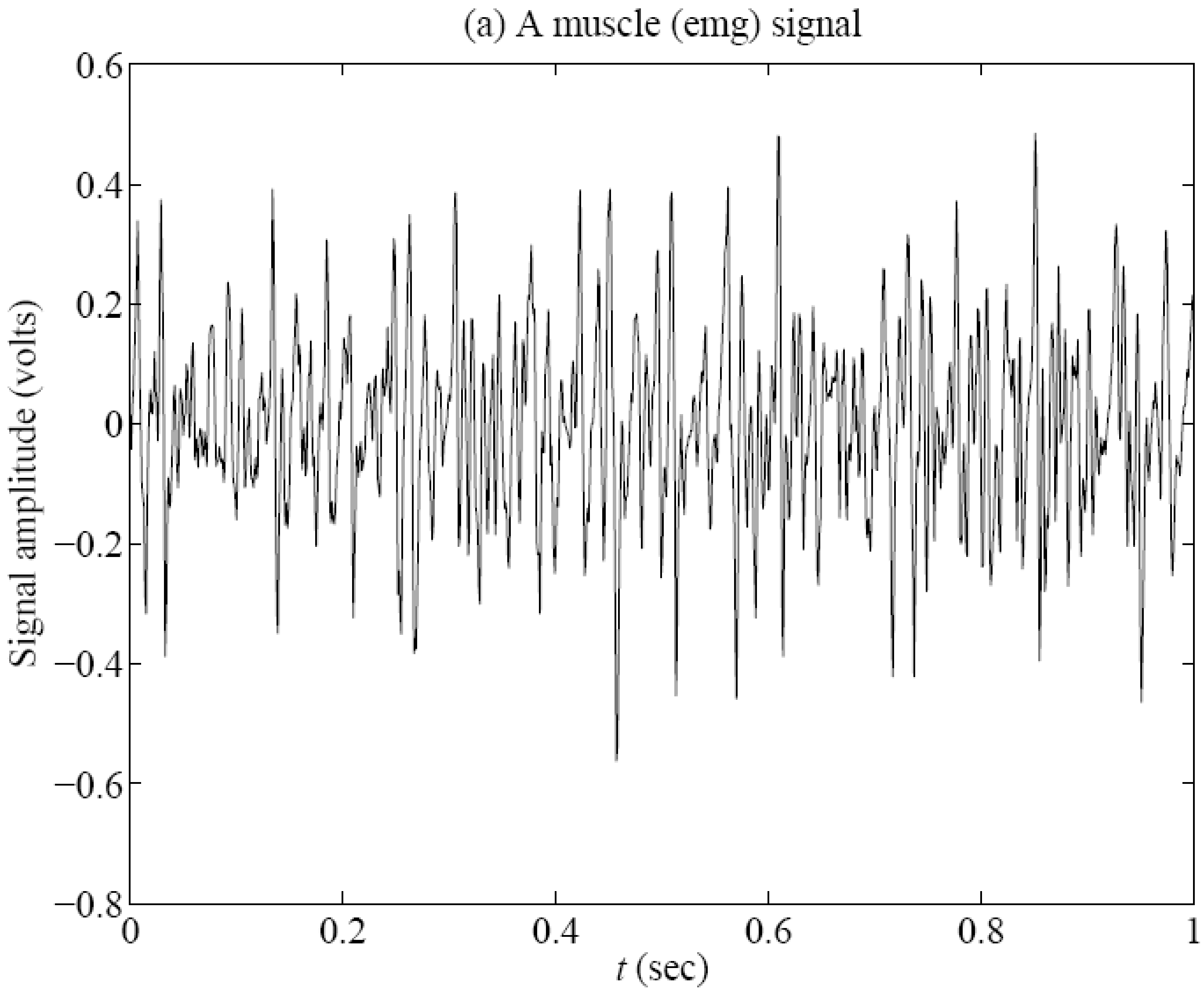
- Variance:

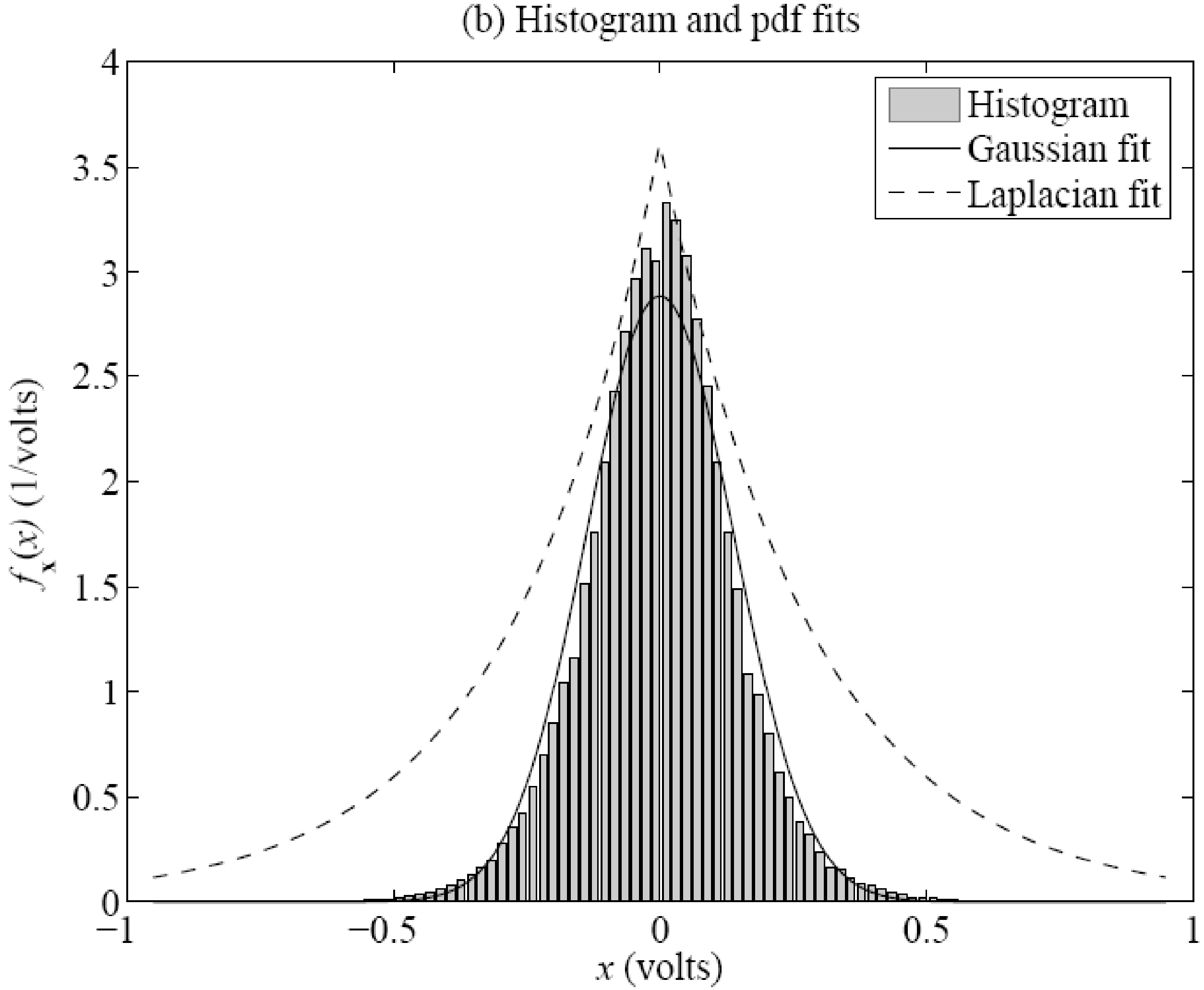
$$\sigma_x^2 = \int_{-\infty}^{\infty}(x-5)^2 \cdot p(x)dx = \int_0^{10}(x-5)^2 \cdot \frac{1}{10}dx = \frac{25}{3}$$

- Probability Calculation:

$$P(6 \le x \le 9) = \int_6^9 p(x)dx = \int_6^9 \frac{1}{10}dx = 0.3$$

The Gaussian Random Variable

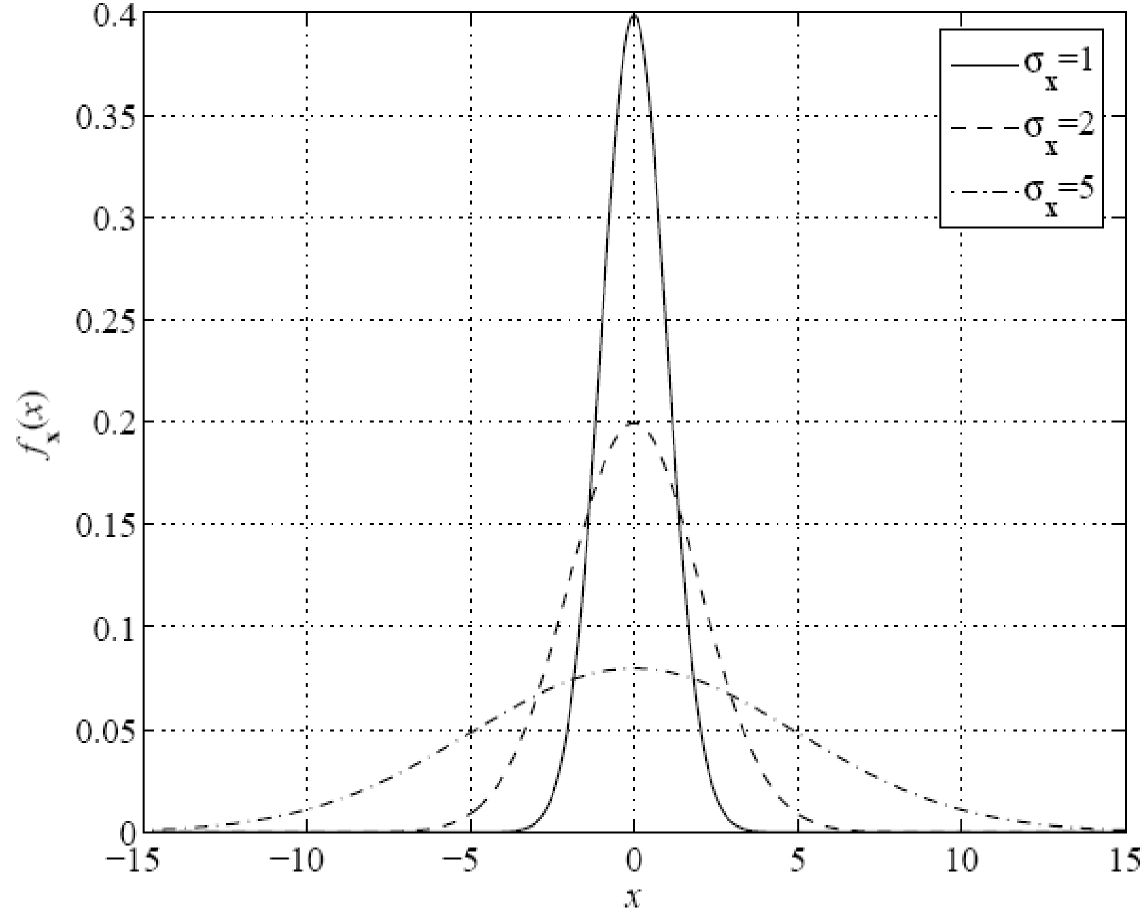




$$p(x) = \frac{1}{\sqrt{2\pi\sigma_{\mathbf{x}}^2}} e^{-\frac{(x-m_{\mathbf{x}})^2}{2\sigma_{\mathbf{x}}^2}} \quad \text{(Gaussian)}$$

$$p(x) = \frac{a}{2} e^{-a|x|} \quad \text{(Laplacian)}$$

Gaussian Distribution (Univariate)



Range ($\pm k\sigma_{\mathbf{x}}$)	$k=1$	$k=2$	$k=3$	$k=4$
$P(m_{\mathbf{x}} - k\sigma_{\mathbf{x}} < \mathbf{x} \leq m_{\mathbf{x}} - k\sigma_{\mathbf{x}})$	0.683	0.955	0.997	0.999
Error probability	10^{-3}	10^{-4}	10^{-6}	10^{-8}
Distance from the mean	3.09	3.72	4.75	5.61