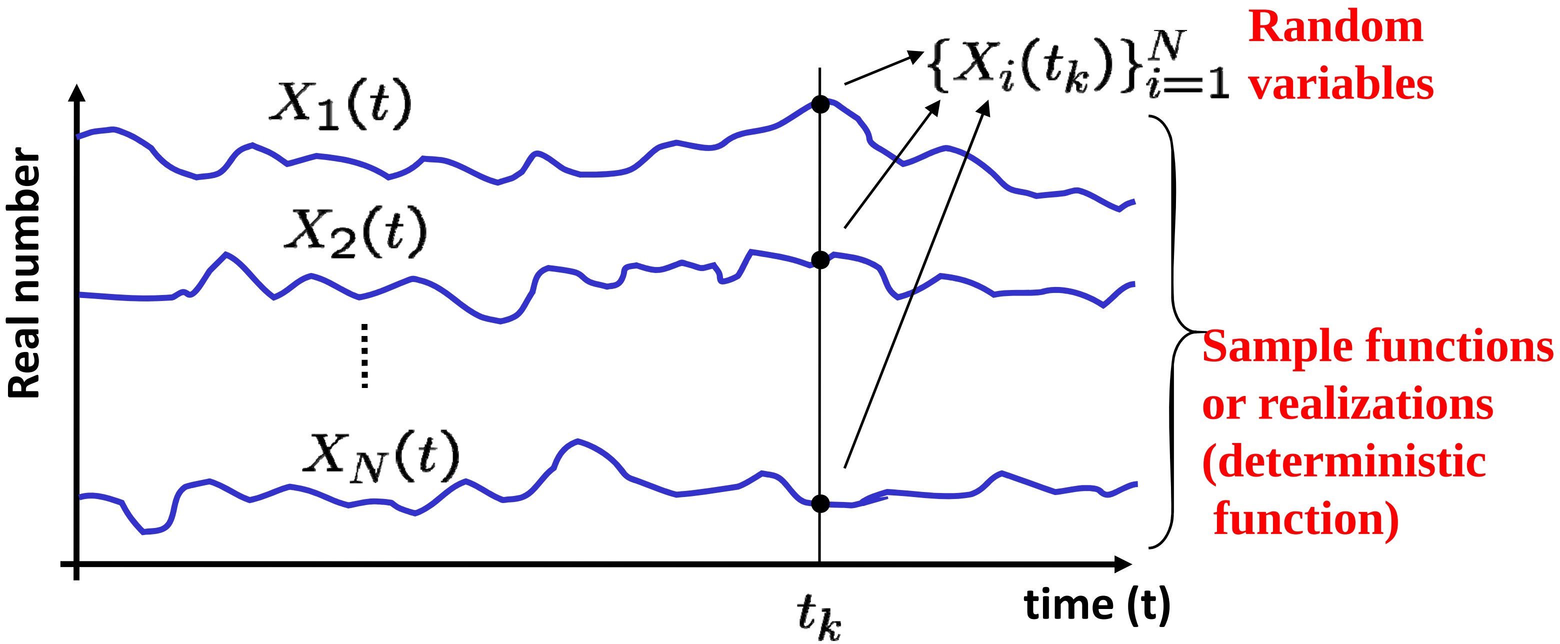
1.5.2 Random Processes

A random process $X(A, t)$ can be viewed as a function of two variables: *an event A* and *time*. Figure 1.5 illustrates a random process. In the figure there are N *sample functions* of time, $\{X_j(t)\}$. Each of the sample functions can be regarded as the output of a different noise generator. For a specific event A_j, we have a single time function $X(A_j, t) = X_j(t)$ (i.e., a sample function). The totality of all sample functions is called an *ensemble*. For a specific time t_k, $X(A, t_k)$ is a *random variable* $X(t_k)$ whose value depends on the event. Finally, for a specific event, $A = A_j$ and a specific time $t = t_k$, $X(A_j, t_k)$ is simply a *number*. For notational convenience we shall designate the random process by $X(t)$, and let the functional dependence upon A be implicit.

Random process

- A random process is a collection of time functions, or signals, corresponding to various outcomes of a random experiment. For each outcome, there exists a deterministic function, which is called a sample function or a realization.



1.5.2.1 Statistical Averages of a Random Process

the mean and autocorrelation function are often adequate for the needs of communication systems.

the mean of the random process $X(t)$ as

$$\mathbf{E}\{X(t_k)\} = \int_{-\infty}^{\infty} x p_{X_k}(x)\, dx = m_X(t_k)$$

where $X(t_k)$ is the random variable obtained by observing the random process at time t_k and the pdf of $X(t_k)$, the density over the ensemble of events at time t_k, is designated $p_{X_k}(x)$.

We define the autocorrelation function of the random process $X(t)$ to be a function of two variables, t_1 and t_2, given by

$$R_X(t_1, t_2) = \mathbf{E}\{X(t_1)X(t_2)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

$f_{X(t_1),X(t_2)}(x_1, x_2)$ is the second-order probability density function of the process

The joint probability density functionof the random variables x(t1) and x(t2)

1.5.2.2 Stationarity

A random process $X(t)$ is said to be *stationary* in the *strict sense* if none of its statistics are affected by a shift in the time origin. A random process is said to be *wide-sense stationary* (WSS) if two of its statistics, its mean and autocorrelation function, do not vary with a shift in the time origin. Thus, a process is WSS if

$$\mathbf{E}\{X(t)\} = m_X = \text{a constant} \tag{1.32}$$

and

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \tag{1.33}$$

1.5.2.3 Autocorrelation of a Wide-Sense Stationary Random Process

Just as the variance provides a measure of randomness for random variables, the autocorrelation function provides a similar measure for random processes. For a wide-sense stationary process, the autocorrelation function is only a function of the *time difference* $\tau = t_1 - t_2$; that is,

$$R_X(\tau) = \mathbf{E}\{X(t)X(t+\tau)\} \quad \text{for } -\infty < \tau < \infty \tag{1.34}$$

Summary : Autocorrelation

- Autocorrelation of an energy signal

$$R_x(\tau) = x(\tau) \star x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a power signal

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

  - For a periodic signal:

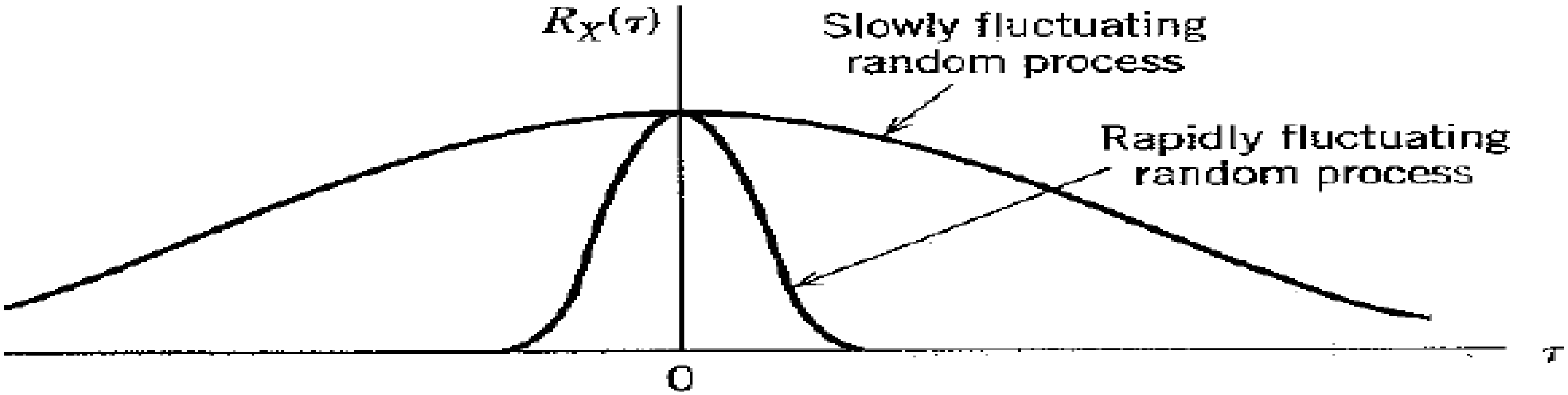
$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a random signal

  - For a WSS process: $R_X(t_i, t_j) = \mathsf{E}[X(t_i)X^*(t_j)]$

$$R_X(\tau) = \mathsf{E}[X(t)X^*(t-\tau)]$$

1. $R_X(\tau) = R_X(-\tau)$ — symmetrical in τ about zero
2. $R_X(\tau) \leq R_X(0)$ for all τ — maximum value occurs at the origin
3. $R_X(\tau) \leftrightarrow G_X(f)$ — autocorrelation and power spectral density form a Fourier transform pair
4. $R_X(0) = \mathbf{E}\{X^2(t)\}$ — value at the origin is equal to the average power of the signal



RE 5.8 Illustrating the autocorrelation functions of slowly
pidly fluctuating random processes.

1.5.3 Time Averaging and Ergodicity

To compute m_X and $R_X(\tau)$ by ensemble averaging, we would have to average across all the sample functions of the process and would need to have complete knowledge of the first- and second-order joint probability density functions. Such knowledge is generally not available.

When a random process belongs to a special class, known as an *ergodic process*, its time averages equal its ensemble averages, and the statistical properties of the process can be determined by *time averaging over a single sample function* of the process. For a random process to be ergodic, it must be stationary in the strict sense. (The converse is not necessary.) However, for communication systems, where we are satisfied to meet the conditions of wide-sense stationarity, we are interested only in the mean and autocorrelation functions.

We can say that a random process is *ergodic in the mean* if

$$m_X = \lim_{T\to\infty} 1/T \int_{-T/2}^{T/2} X(t)\, dt \tag{1.35}$$

and it is *ergodic in the autocorrelation function* if

$$R_X(\tau) = \lim_{T\to\infty} 1/T \int_{-T/2}^{T/2} X(t)X(t+\tau)\, dt \tag{1.36}$$

*We use **time averaging** instead of **ensemble averaging*** averages equal ensemble averages for ergodic processes, fundamental electrical engineering parameters, such as dc value, rms value, and average power can be related to the moments of an ergodic random process. Following is a summary of

1. The quantity $m_X = \mathbf{E}\{X(t)\}$ is equal to the dc level of the signal.
2. The quantity m_X^2 is equal to the normalized power in the dc component.
3. The second moment of $X(t)$, $\mathbf{E}\{X^2(t)\}$, is equal to the total average normalized power.
4. The quantity $\sqrt{\mathbf{E}\{X^2(t)\}}$ is equal to the root-mean-square (rms) value of the voltage or current signal.
5. The variance σ_X^2 is equal to the average normalized power in the time-varying or ac component of the signal.
6. If the process has zero mean (i.e., $m_X = m_X^2 = 0$), then $\sigma_X^2 = \mathbf{E}\{X^2\}$ and the variance is the same as the mean-square value, or the variance represents the total power in the normalized load.
7. The standard deviation σ_X is the rms value of the ac component of the signal.
8. If $m_X = 0$, then σ_X is the rms value of the signal.

$$\sigma_X^2 = E\{x^2\} - [E\{x\}]^2 = E\{x^2\} - m_X^2.$$

1.5.4 Power Spectral Density and Autocorrelation of a Random Process

A random process $X(t)$ can generally be classified as a power signal having a power spectral density (PSD) $G_X(f)$ of the form shown in Equation (1.20). $G_X(f)$ is particularly useful in communication systems, because it describes the distribution of a signal's power in the frequency domain. The PSD enables us to evaluate the signal power that will pass through a network having known frequency characteristics. We summarize the principal features of PSD functions as follows:

1.	$G_X(f) \geq 0$	and is always real valued
2.	$G_X(f) = G_X(-f)$	for $X(t)$ real-valued
3.	$G_X(f) \leftrightarrow R_X(\tau)$	PSD and autocorrelation form a Fourier transform pair
4.	$P_X = \int_{-\infty}^{\infty} G_X(f)\, df$	relationship between average normalized power and PSD

Summary : Random process …

- **Strictly stationary:** If none of the statistics of the random process are affected by a shift in the time origin.
- **Wide sense stationary (WSS):** If the mean and autocorrelation function do not change with a shift in the origin time.
- **Cyclostationary:** If the mean and autocorrelation function are periodic in time.
- **Ergodic process:** A random process is ergodic in mean and autocorrelation, if

$$m_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)dt$$

and

$$R_X(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X^*(t-\tau)dt$$

Numerical Example

$X(t)$ Random binary sequence

sample waveform from a WSS random process, $X(t)$.

$X(t-\tau_1)$

same sequence displaced τ_1 seconds in time;

$$R_X(\tau_1) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,X(t-\tau_1)\,dt$$

$X(t)$ and $X(t-\tau_1)$ and finding the average value

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & \text{for } |\tau| < T \\ 0 & \text{for } |\tau| > T \end{cases}$$

$$G_X(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

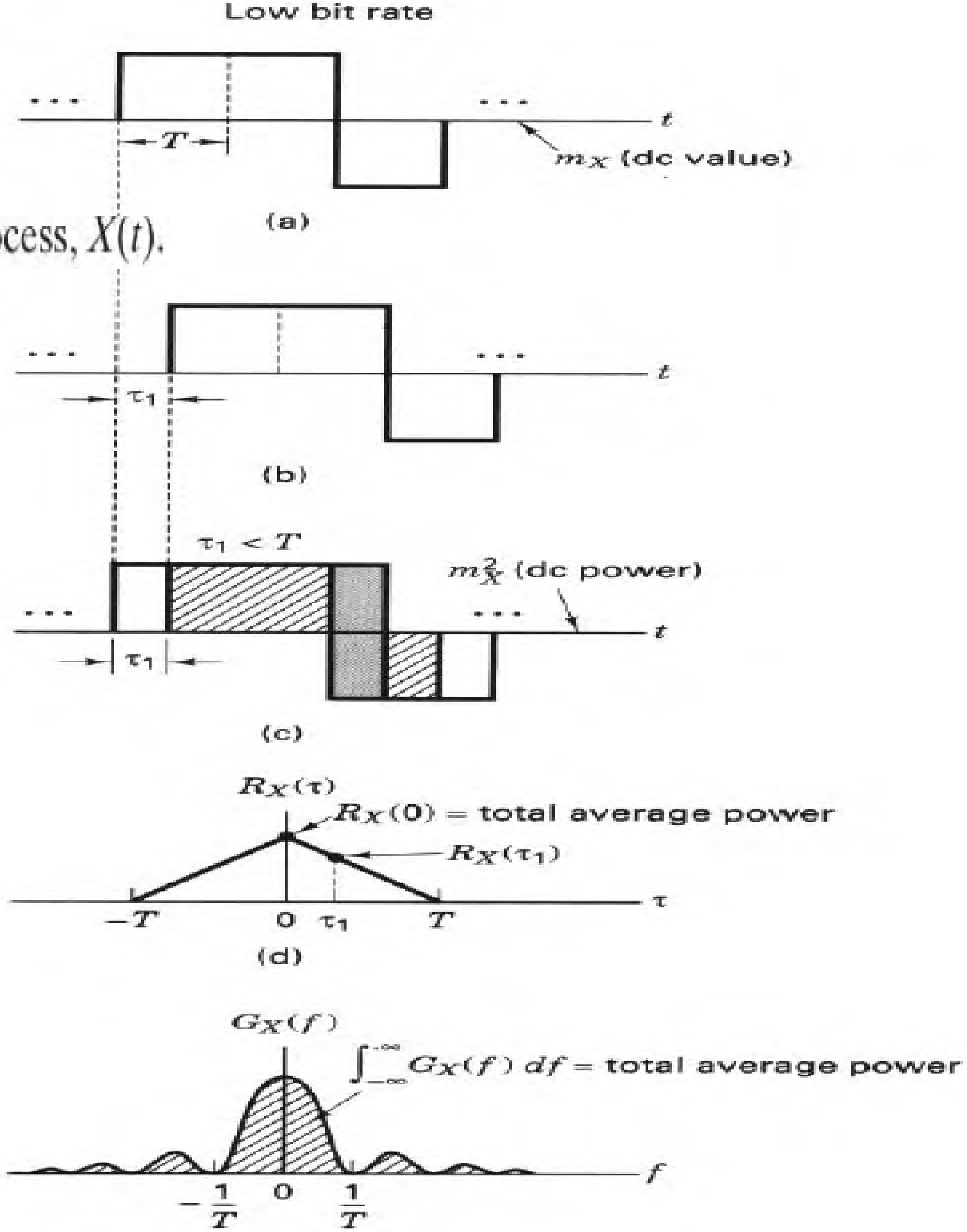


Figure 1.6 Autocorrelation and power spectral density.

the pulse duration is shorter

$X(t)$ Random binary sequence

$X(t-\tau_1)$

$$R_X(\tau_1) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,X(t-\tau_1)\,dt$$

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & \text{for } |\tau| < T \\ 0 & \text{for } |\tau| > T \end{cases}$$

$$G_X(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

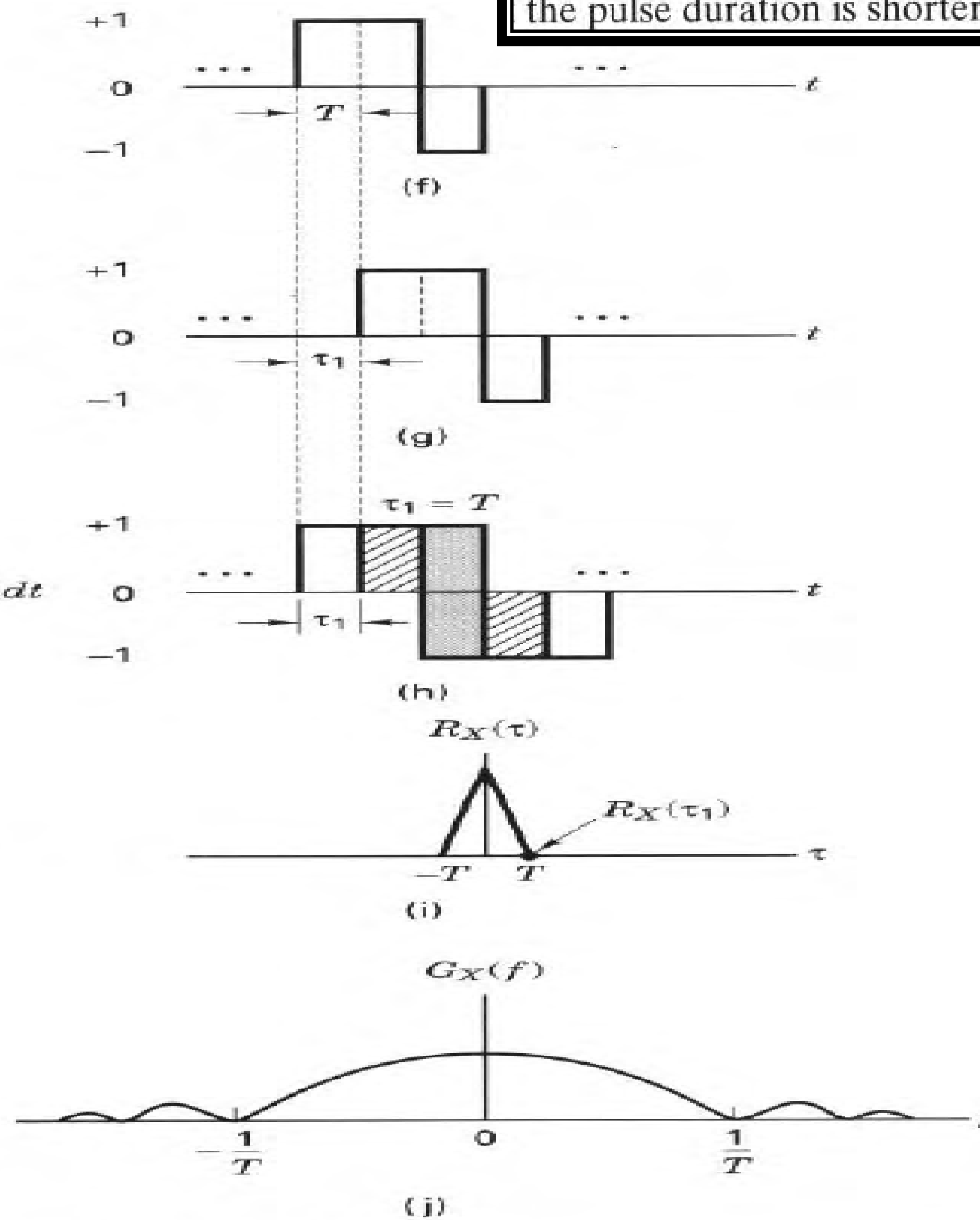


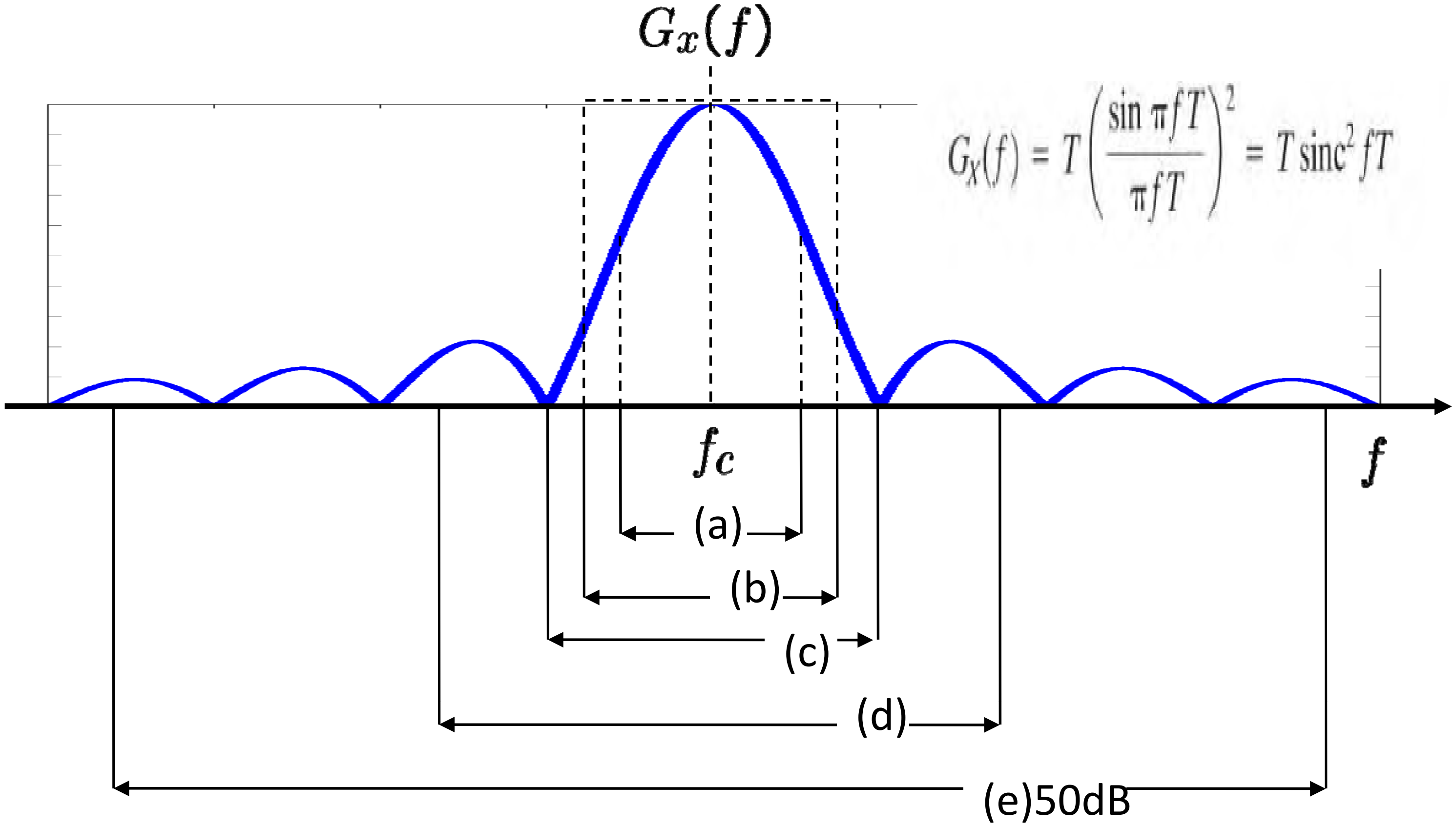
Figure 1.6 continued

Bandwidth of signal …

- Different definition of bandwidth:

a) Half-power bandwidth
b) Noise equivalent bandwidth
c) Null-to-null bandwidth

a) Fractional power containment bandwidth
b) Bounded power spectral density
c) Absolute bandwidth



$$G_X(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2 = T \operatorname{sinc}^2 fT \qquad (1.38)$$

(a) *Half-power bandwidth.* This is the interval between frequencies at which $G_x(f)$ has dropped to half-power, or 3 dB below the peak value.

(b) *Equivalent rectangular or noise equivalent bandwidth.* The noise equivalent bandwidth was originally conceived to permit rapid computation of output noise power from an amplifier with a wideband noise input; the concept can similarly be applied to a signal bandwidth. The noise equivalent bandwidth

W_N of a signal is defined by the relationship $W_N = P_x/G_x(f_c)$, where P_x is the total signal power over all frequencies and $G_x(f_c)$ is the value of $G_x(f)$ at the band center (assumed to be the maximum value over all frequencies).

(c) *Null-to-null bandwidth.* The most popular measure of bandwidth for digital communications is the width of the main spectral lobe, where most of the signal power is contained. This criterion lacks complete generality since some modulation formats lack well-defined lobes.

(d) *Fractional power containment bandwidth.* This bandwidth criterion has been adopted by the Federal Communications Commission (FCC Rules and Regulations Section 2.202) and states that the occupied bandwidth is the band that leaves exactly 0.5% of the signal power above the upper band limit and exactly 0.5% of the signal power below the lower band limit. Thus 99% of the signal power is inside the occupied band.

(e) *Bounded power spectral density.* A popular method of specifying bandwidth is to state that everywhere outside the specified band, $G_x(f)$ must have fallen at least to a certain stated level below that found at the band center. Typical attenuation levels might be 35 or 50 dB.

(f) *Absolute bandwidth.* This is the interval between frequencies, outside of which the spectrum is zero. This is a useful abstraction. However, for all realizable waveforms, the absolute bandwidth is infinite.

1.12 The power spectral density of a random process $X(t)$ is shown in Figure P1.12. It consists of a delta function at $f = 0$ and a triangular component.

(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.

(b) What is the DC power contained in $X(t)$?

(c) What is the AC power contained in $X(t)$?

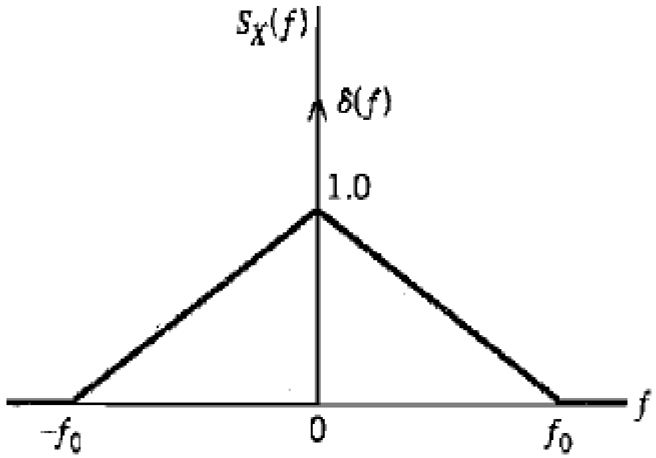


FIGURE P1.12

(a) The power spectral density consists of two components:

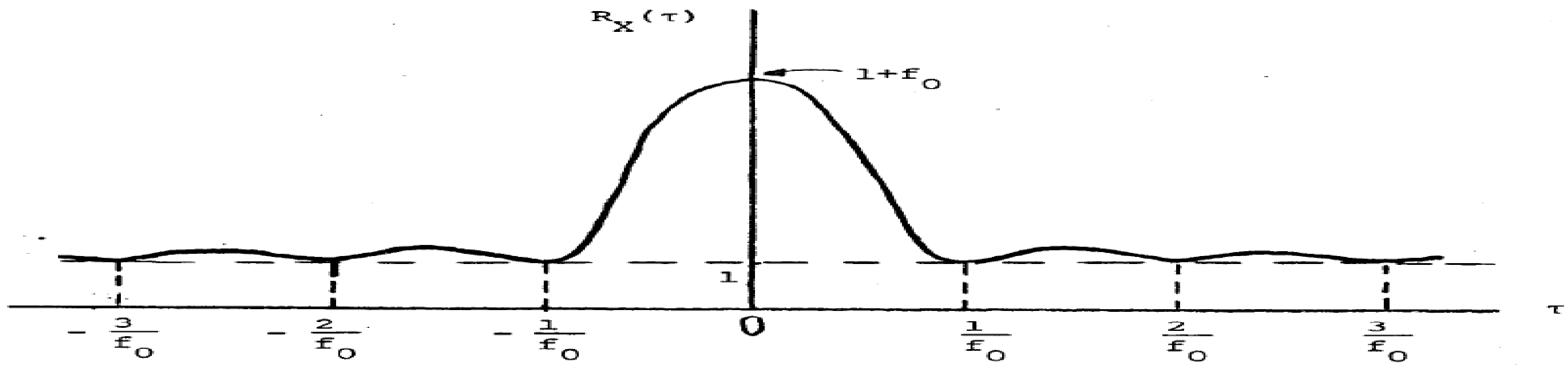
(1) A delta function $\delta(t)$ at the origin, whose inverse Fourier transform is one.

(2) A triangular component of unit amplitude and width $2f_0$, centered at the origin; the inverse Fourier transform of this component is $f_0 \operatorname{sinc}^2(f_0\tau)$.

Therefore, the autocorrelation function of X(t) is

$$R_X(\tau) = 1 + f_0 \operatorname{sinc}^2(f_0\tau)$$

which is sketched below:



(b) Since $R_X(\tau)$ contains a constant component of amplitude 1, it follows that the dc power contained in X(t) is 1.

(c) The mean-square value of X(t) is given by

$$E[X^2(t)] = R_X(0)$$
$$= 1 + f_0$$

The ac power contained in X(f) is therefore equal to f_0.