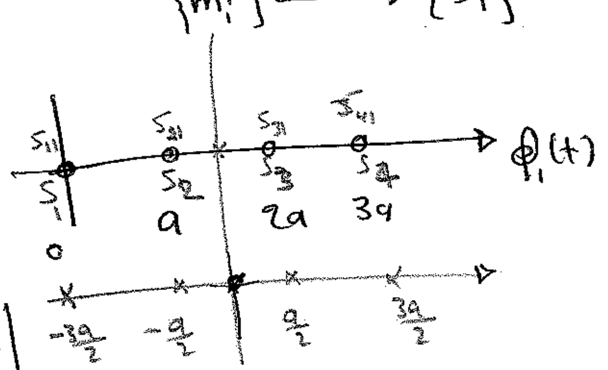
(0)

① Minimum Energy signals (Translation)

$\{m_i\} \longrightarrow \{s_i\}$

assume equally a periori signals



$$E_{av} = \frac{0 + (a)^2 + (2a)^2 + (3a)^2}{4}$$

$$\boxed{E_{av}} = \frac{a^2 + 4a^2 + 9a^2}{4} = \boxed{\frac{14}{4} a^2}$$

Translate the coordinates to the Center of the signals Constellation

$$E_{av}\Big|_{min} = \frac{2\times(\frac{a}{2})^2 + 2\times(\frac{3a}{2})^2}{4} = \frac{\frac{a^2}{2} + \frac{9a^2}{2}}{4} = \frac{\overset{5}{\cancel{10}} a^2}{\cancel{2}}$$

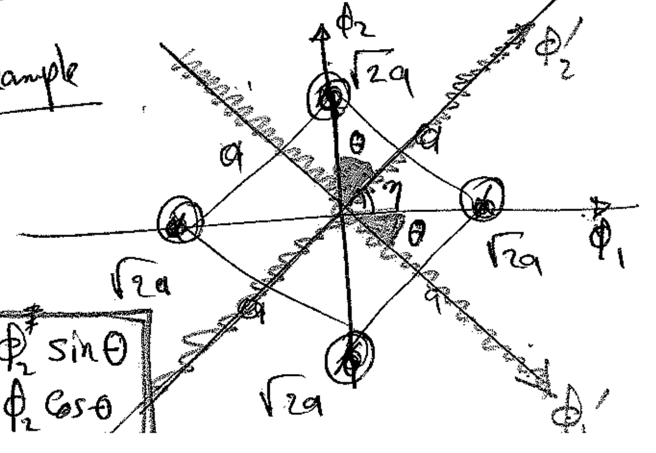
$$\boxed{E_{av}\Big|_{min}} = \boxed{\frac{5a^2}{4}}$$

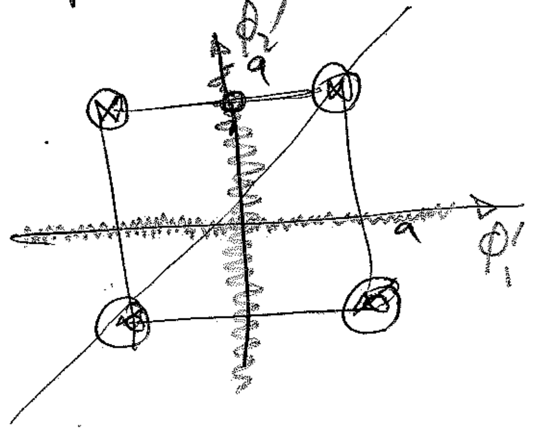
by translation we obtain 4 signals with minimum average energy and the same probability of error since the distance between ~~dem~~ signals are unchanged.

② Rotation

also doesn't affect P_e as P_e depends on $\|X - s_i\|$

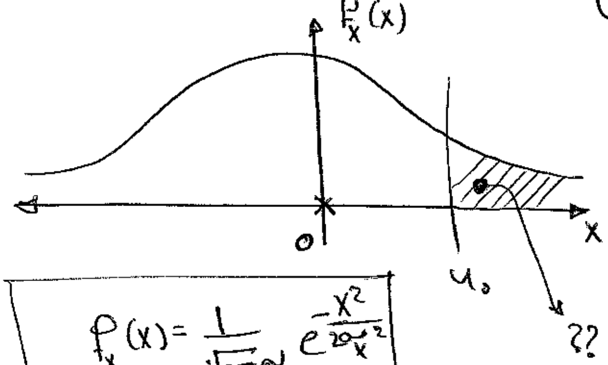
Example





$$\phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta$$
$$\phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta$$

Pass band Data Transmission



$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-z^2} dz$$

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-x^2}{2\sigma_X^2}}}$$

$$erfc(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-z^2} dz \longrightarrow (1)$$

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{z^2}{2}} dz \longrightarrow (2)$$

$erfc(u) = 1 - erf(u)$

$erf(\infty) = 1 \quad erfc(\infty) = 0$

$$\therefore \boxed{Q(u) = \frac{1}{2} erfc\left(\frac{u}{\sqrt{2}}\right)} \longrightarrow (3)$$

$$\boxed{erfc(u) = 2Q(\sqrt{2}u)} \longrightarrow (4)$$

* the integral of the Gaussian pdf zeromean & variance $= \sigma^2$ from $u_0 \longrightarrow \infty$ = ?? I

$$I = \frac{1}{\sigma_X\sqrt{2\pi}} \int_{u_0}^\infty e^{-\frac{x^2}{2\sigma_X^2}} dX \longrightarrow (5)$$

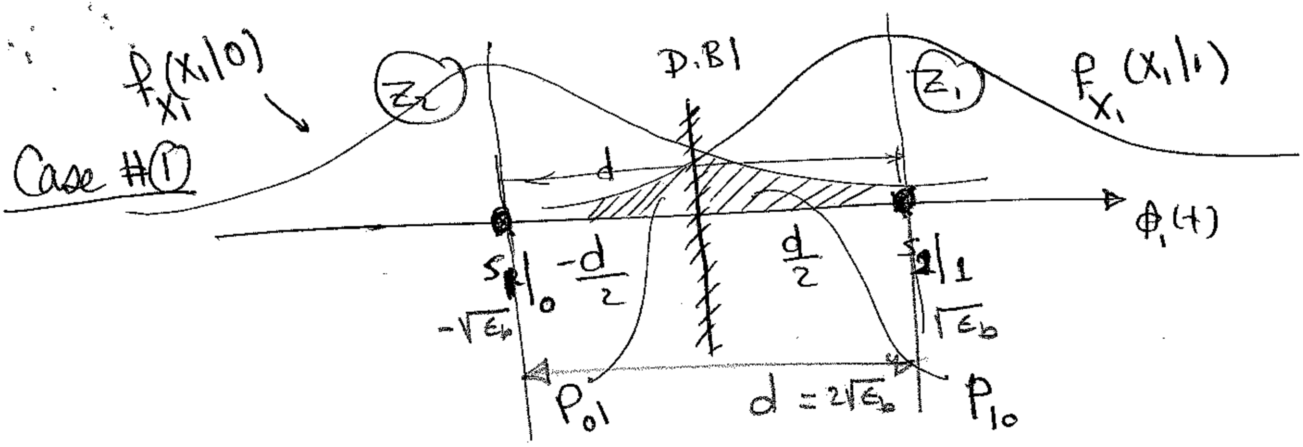
Compare (5) wz (2) assume $z \longrightarrow \frac{X}{\sigma_X}$ then $u \longrightarrow \frac{u_0}{\sigma_X}$

$$\therefore \boxed{I = Q\left(\frac{u_0}{\sigma_X}\right)}$$

if pdf Gaussian distribution with variance $= \frac{N_0}{2} = \sigma_X^2$

$$\therefore \boxed{I = Q\left(\frac{u_0}{\sqrt{N_0/2}}\right)}$$

Case #1



$$X = S_i + W$$

$$X_j = S_{ij} + W_j \qquad j = 1, 2, \dots, N \text{ (for all } i)$$

X_j → r.v; S_{ij} → deterministic; W_j → r.v

$$\boxed{j = 1 \quad \phi_1(t) \text{ only}}$$

$$\therefore X_1 = \underset{+\frac{d}{2} \quad -\frac{d}{2}}{S_{i1}} + W_1 \longrightarrow GN(0, \sigma_n^2) \qquad \sigma_n^2 = \boxed{\frac{N_0}{2}}$$

① D.B1 $\boxed{x_1 = 0}$

D.Rs: $Z_1 : x_1 > 0$

$Z_2 : x_1 < 0$

② the average transmitted energy $= E_{av} = \frac{\cancel{2} \times \left(\frac{d}{2}\right)^2}{\cancel{2}} = \frac{d^2}{4}$ $= \left(\sum s_{ij}^2\right)$

that is the minimum energy

③ the average P_e noise AWGN zero mean & (PSD $= \frac{N_0}{2}$)

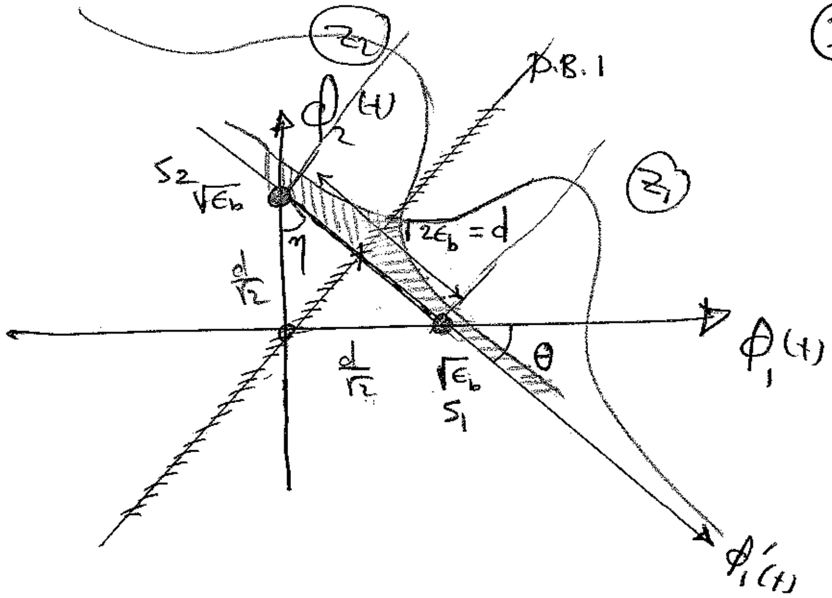
$$P_e = \underbrace{P(s_1)}_{\frac{1}{2}} P_{10} + \underbrace{P(s_2)}_{\frac{1}{2}} P_{01} \qquad P_{01} = P_{10}$$

BPSK

$$P_e = P_{01} = Q\left(\frac{\frac{d}{2}}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right) = \boxed{Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

General ($d = 2\sqrt{E_b}$)

Case # 2



[1] the D.B $\Rightarrow$ $\theta = 90°$

Z_1 & Z_2 as shown

$$\phi_1(t) \;\&\; \phi_2(t) \xrightarrow[\text{of axes}]{\text{Rotation}} \phi_1'(t)$$

in order to obtain a single arm Rx, Rotation of axes must be performed, by angle θ (clockwise)

From S.S: $$\eta° = \tan^{-1}\left(\frac{d/\sqrt{2}}{d/\sqrt{2}}\right) = 45°$$

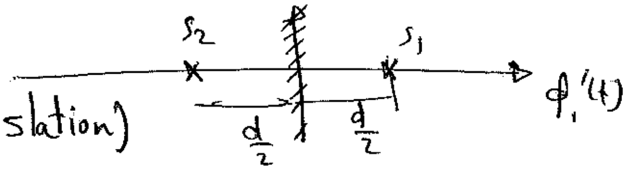
$$\therefore \theta° = 90° - \eta° = 45°$$

$$\phi_1'(t) = \phi_1(t)\cos\theta - \phi_2(t)\sin\theta$$

$$\boxed{\phi_1'(t) = \frac{1}{\sqrt{2}}\phi_1(t) - \frac{1}{\sqrt{2}}\phi_2(t)}$$

[2] The average transmitted energy $= E_{av} = \dfrac{2 \times \left(\frac{d^2}{2} + \frac{d^2}{2}\right)}{2} = d^2$

[3] equivalent set of messages with the same probability of error but w/ $E_{av}\big|_{min}$ (by translation)



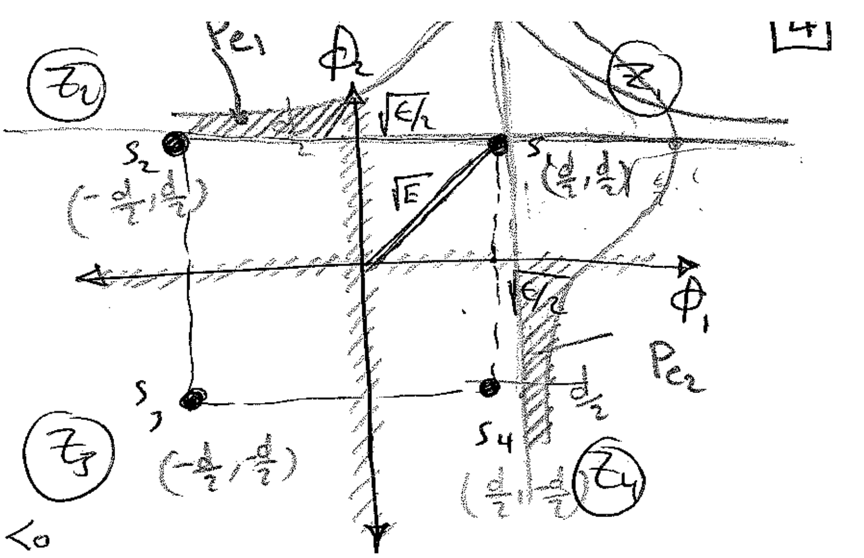
$$E_{av} = \frac{2 \times \left(\frac{d}{2}\right)^2}{2} = \frac{d^2}{4}$$

if $d = \sqrt{2\mathcal{E}_b}$

[4] symbol error

$$P_e = P_{S_1} P_{10} + P_{S_2} P_{01} = \left\{ P_{01} = Q\left(\frac{d}{\sqrt{2N_0}}\right) \right\} \quad \therefore \boxed{P_e = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)}$$

Case (3)



Four message points s_1, s_2, s_3, s_4

$X_j = s_{ij} + W_j$

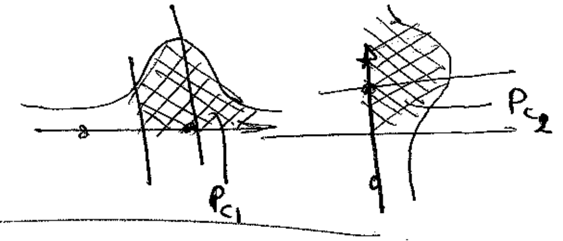
$\therefore \boxed{\begin{array}{l} X_1 = s_{i1} + W_1 \\ X_2 = s_{i2} + W_2 \end{array}}$

(1) the D.R1 $X_2 = 0$

D.R2 $X_1 = 0$

(Z_1): $X_1 > 0$, $X_2 > 0$ $\qquad$ Z_3: $X_1 < 0$, $X_2 < 0$

(Z_2) $X_1 < 0$, $X_2 > 0$ $\qquad$ Z_4: $X_1 > 0$, $X_2 < 0$



(2) the average energy $= E_{av} = \frac{1}{4}\left(\sum_{i=1}^{4} E_i\right)$

$$= \frac{1}{4}\Big(4 \times \underbrace{\left(\sqrt{\tfrac{E}{2}}\right)^2 + \left(\sqrt{\tfrac{E}{2}}\right)^2}_{\left(\frac{E}{2}\right)}\Big) = \frac{E}{2}$$

$$= \frac{1}{4}\left(4\left[\left(\tfrac{d}{2}\right)^2 + \left(\tfrac{d}{2}\right)^2\right]\right) = \boxed{\frac{d^2}{2}}$$

(3) $\boxed{E_{av} = E_{min}}$

(4) the minimum average P_e: (AWGN, 0 mean & $\frac{N_0}{2}$ variance)

X_1 & X_2 are statistically independent $\qquad P(A,B) = P(A)P(B)$

simultaneously

$$P(c|s_1) = P(X_1 > 0) * P(X_2 > 0)$$

if $P(c|s_1) = P(X_1<0) \cdot P(X_2<0)$

$P(X_1<0) \cdot P(X_2>0)$

$P(X_1>0) \cdot P(X_2<0)$

$\therefore$ the average probability of a correct decision resulting from Combined action of the two channels working together

$$P(c|s_1) = (1 - P_{e_1}) \cdot (1 - P_{e_2})$$

$2\sqrt{\frac{E}{2}}$

$$P_{e_1} = P_{e_2} = \frac{1}{\sigma_x\sqrt{2\pi}}\int_0^{d/2} e^{-\frac{x^2}{2\sigma_x^2}}\,dx = Q\left(\frac{d/2}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$P(c|s_1) = (1 - P_e)^2 = 1 - 2P_e - P_e^2$$

$$\therefore\; P_{av}[c] = \cancel{\tfrac{1}{4}}\left(\cancel{4}\left(1 - 2P_e + P_e^2\right)\right)$$

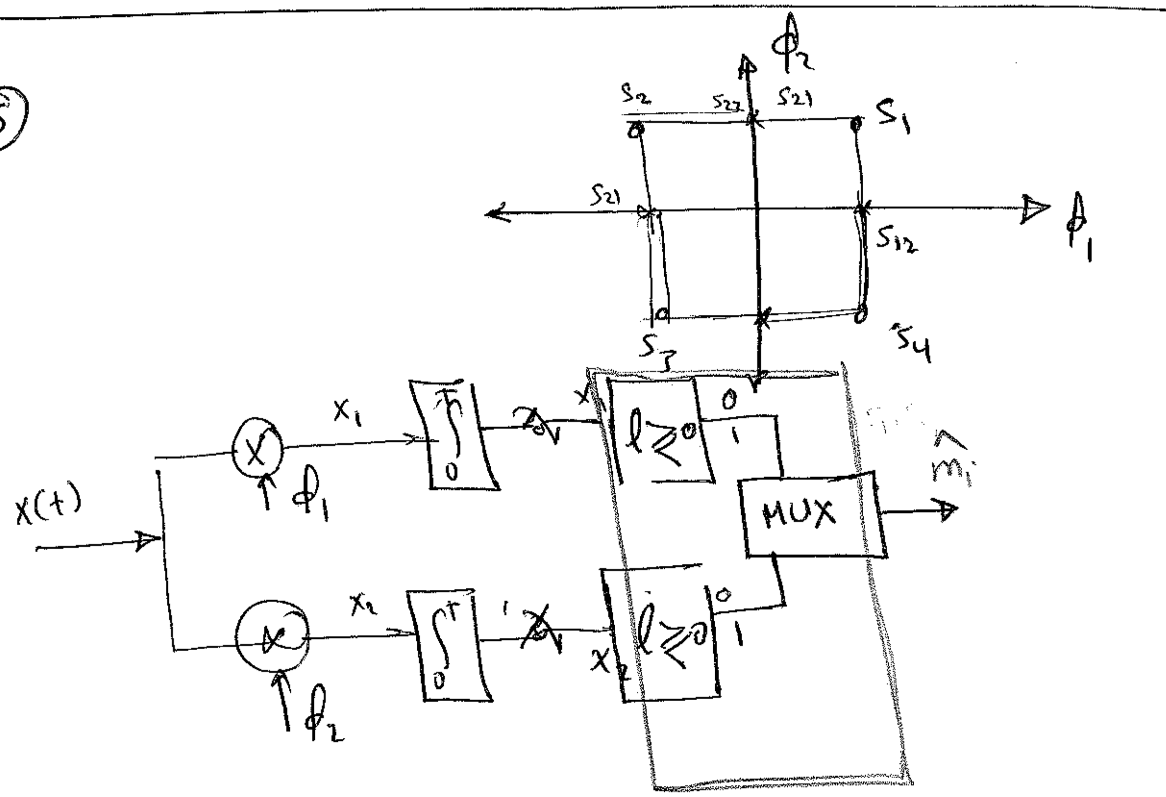
$$P_{av}[\epsilon] = 1 - P_{av}[c] = 2P_e - P_e^2$$

$$\therefore\; P_{av}(\epsilon) = 2Q\left(\frac{d}{\sqrt{2N_0}}\right) - \cancel{\left[Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]^2} \approx 0$$

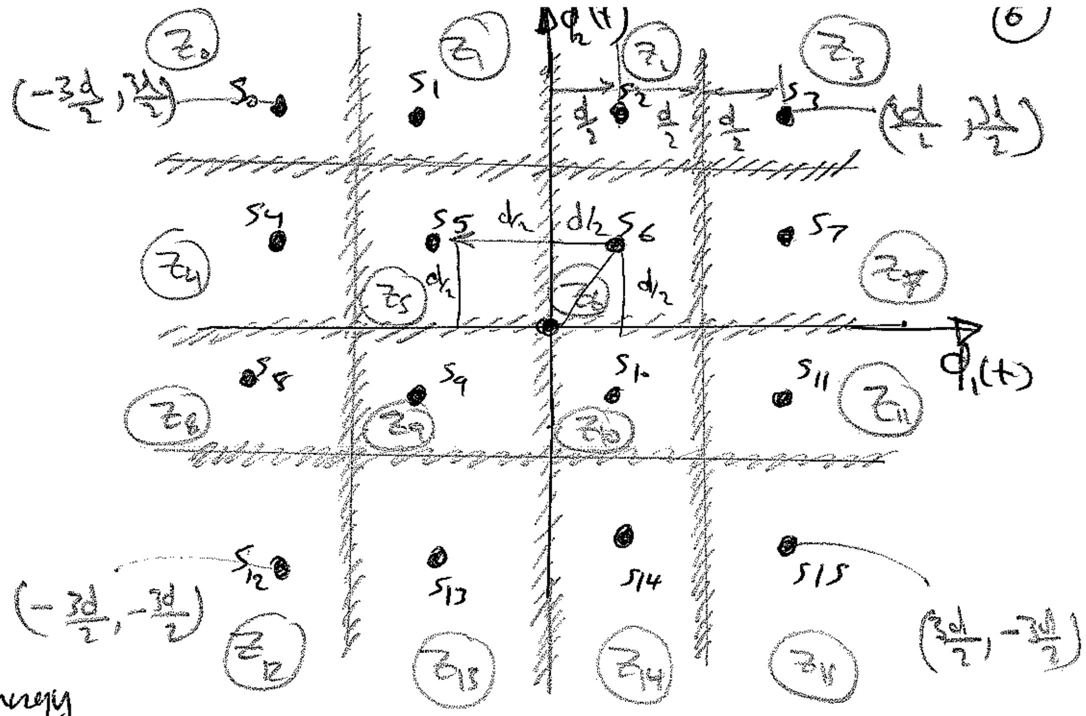
$$P_{av}\Big|_{\text{symbol error}} \approx 2Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$\boxed{P_{av}\Big|_{\text{bit error}} = Q\left(\frac{d}{\sqrt{2N_0}}\right)}$$

5



S.S. Case (4)



(1) B.Bs ✓
D.Rs ✓
as shown.

(2) the average energy
= the minimum energy

$$\Rightarrow E_{av} = \frac{1}{16}\Big[4\times\underbrace{\Big(\frac{d^2}{4}+\frac{d^2}{4}\Big)}_{s_{5,6,9,10}} + 8\times\underbrace{\Big(\frac{9d^2}{4}+\frac{d^2}{4}\Big)}_{s_{1,2,4,11,13,7,8,14}} + 4\times\underbrace{\Big(\frac{9d^2}{4}+\frac{9d^2}{4}\Big)}_{s_{0,3,12,15}}\Big]$$

$$E_{min}\Big|_{av} = \frac{5d^2}{2}$$

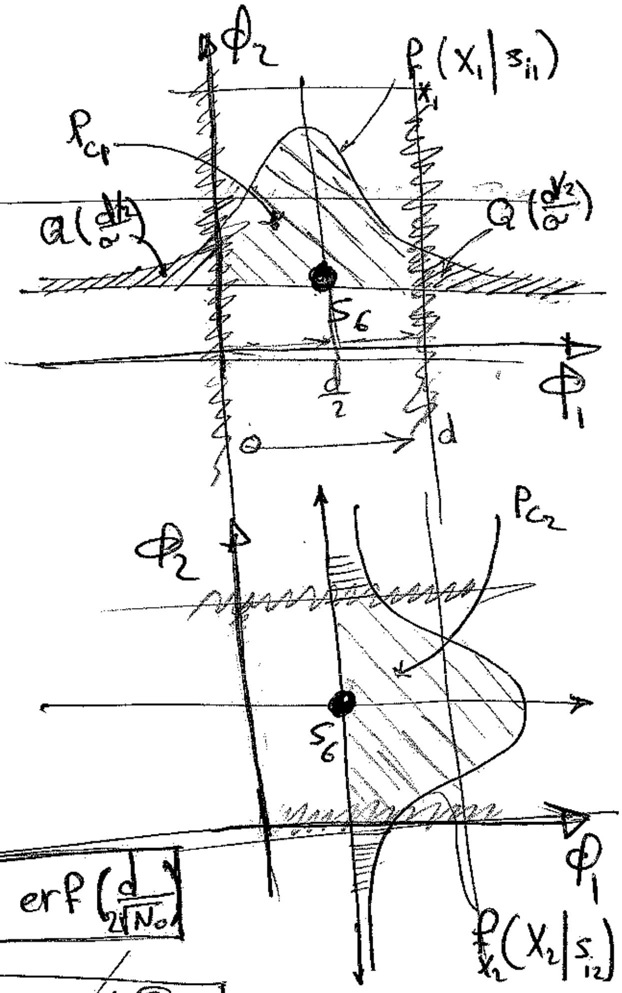
(3) $\therefore d = \sqrt{\frac{2}{5}E}$ for minimum Energy

[4] To find the P_e, it is simpler to first find the probability of Correct P_c of Correct detection for each symbol.

It is clear that we can group signal space points into 3 classes and Computer P_c based on the 3 decisions regions

~~Let~~ X_1, X_2 be ~~AWGN for each~~ Gaussian Samples with mean = S_{ij} & Variance = $\frac{N_o}{2}$ along the ϕ_1 axis and ϕ_2 axis respectively

Case 1: Four inner signal points (s_5, s_6, s_9, s_{10})



$$P(C|s_i) = \underbrace{P\left(-\tfrac{d}{2}<x_1<\tfrac{d}{2}\right)}_{P_c(x_1)} \times \underbrace{P\left(-\tfrac{d}{2}<x_2<\tfrac{d}{2}\right)}_{P_c(x_2)}$$

$$p(c|s_i) = p_{c_1} \times p_{c_2} = p_c^2$$

$$p_{c_1} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\frac{d}{2}}^{d/2} e^{-\frac{x_1^2}{2\sigma^2}} \, dx_1$$

$$= 1 - 2Q\left(\frac{d/2}{\sigma}\right)$$

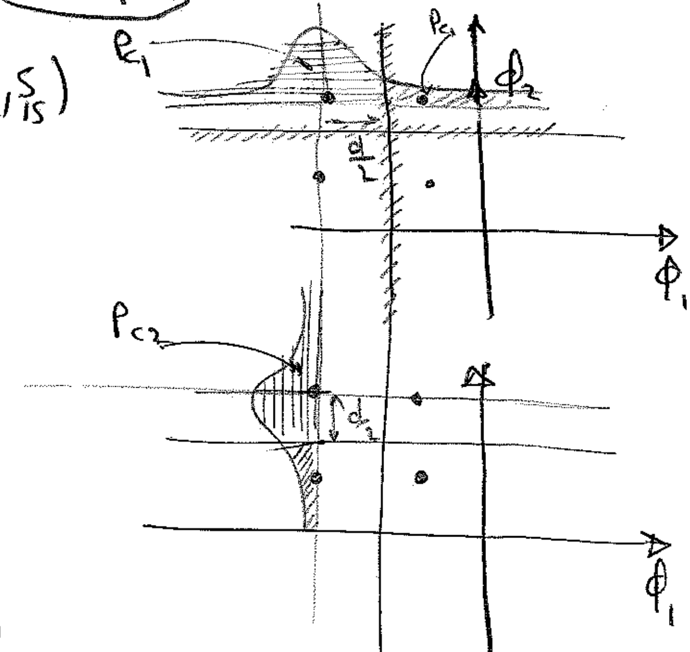
$$= 1 - 2Q\left(\frac{d/2}{\sqrt{\frac{N_0}{2}}}\right)$$

$$p_{c_1} = 1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right) \quad \boxed{= \operatorname{erf}\left(\frac{d}{2\sqrt{N_0}}\right)}$$

$$\therefore \boxed{p(c|s_i) = 1 - 4Q\left(\frac{d}{\sqrt{2N_0}}\right) + 4Q^2\left(\frac{d}{\sqrt{2N_0}}\right)} \Rightarrow$$

$$P(e|s_i)\big|_1 = 1 - P_{c_1} = 4\,Q\left(\frac{d}{\sqrt{2N_0}}\right) \longrightarrow ①$$

Case 2: four corner signal (s_0, s_3, s_{12}, s_{15})



$$P(C|s_i) = \underbrace{P\left(-\infty<x_1<\tfrac{d}{2}\right)}_{P_{c_1}} \times \underbrace{P\left(-\tfrac{d}{2}<x_2<\infty\right)}_{P_{c_2}}$$

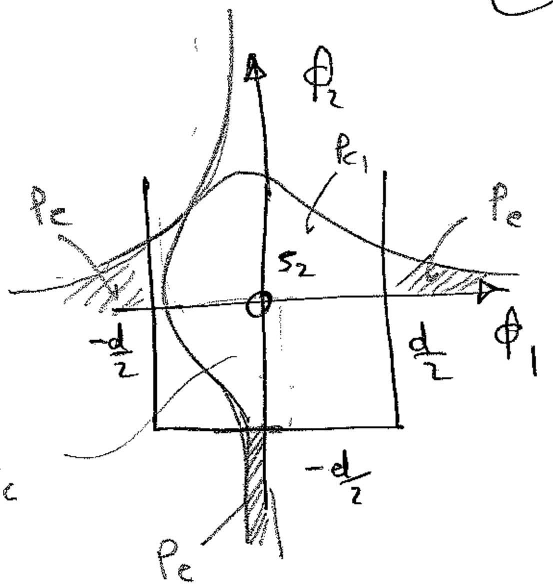
$$= (1 - P_{e_1}) \times (1 - P_{e_2})$$

$$= \left(1 - Q\left(\frac{d/2}{\sigma}\right)\right)\left(1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

$$P(c|s_i) = 1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right) - Q^2\left(\frac{d}{\sqrt{2N_0}}\right) \rightarrow \approx 0$$

$$P(e|s_i)\big|_2 = 1 - P_{c_2} = 2\,Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

Case 3: eight edge signal points (S1, 2, 4, 7, 8, 11, 13, 14)



$$P(c|S_i) =$$

$$= \underbrace{P(-\tfrac{d}{2} < X_1 < \tfrac{d}{2})}_{P_{c_1}} \times \underbrace{P(-\tfrac{d}{2} < X_2 < \infty)}_{P_{c_2}}$$

$$= (1 - P_{e_1}) \times (1 - P_{e_2})$$

$$= \left(1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)\left(1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

$$\therefore P(c|s)_3 = 1 - 3Q\left(\frac{d}{\sqrt{2N_0}}\right) + \cancel{2Q^2\left(\frac{d}{\sqrt{2N_0}}\right)}$$

$$\therefore P_e(\varepsilon|s_i)_3 = 3Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$P_{av}(\varepsilon) = \sum_{i=0}^{15} P(s_i) \times P(\varepsilon|s_i)$$

$$\therefore P_e\Big|_{av} = \cancel{4} \times \left(\frac{1}{\cancel{16}} \times \cancel{4} Q\left(\frac{d}{\sqrt{2N_0}}\right)\right) + \cancel{4}\left(\frac{1}{\cancel{16}} \times \cancel{2} Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

$$+ \cancel{8}\left(\frac{1}{\cancel{16}} \times 3Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)$$

1, $\frac{1}{2}$, $\frac{3}{2}$

$$\boxed{P_e\Big|_{av} = 3Q\left(\frac{d}{\sqrt{2N_0}}\right)}$$

Final answer

$$\Rightarrow \frac{3}{2}\,\mathrm{erfc}\left(\frac{d}{2\sqrt{N_0}}\right)$$

$$d = \sqrt{\frac{2}{5}E}$$

for minimum Energy