Week (6)
Signal-Space Analysis

5.3 Conversion of the Continuous AWGN Channel into a Vector Channel

5.5 Coherent Detection of Signals in Noise: Maximum Likelihood Decoding

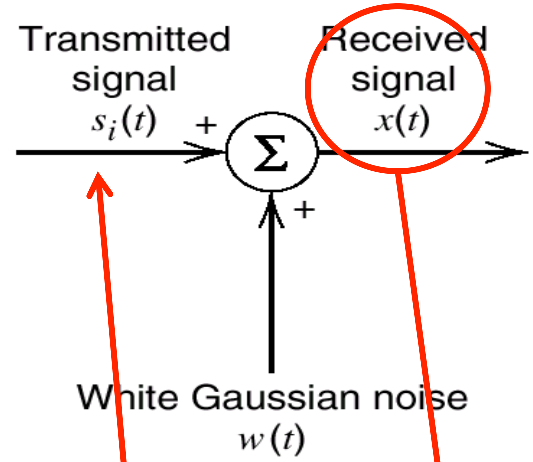


Figure 5.2 Additive white Gaussian noise (AWGN) model of a channel.

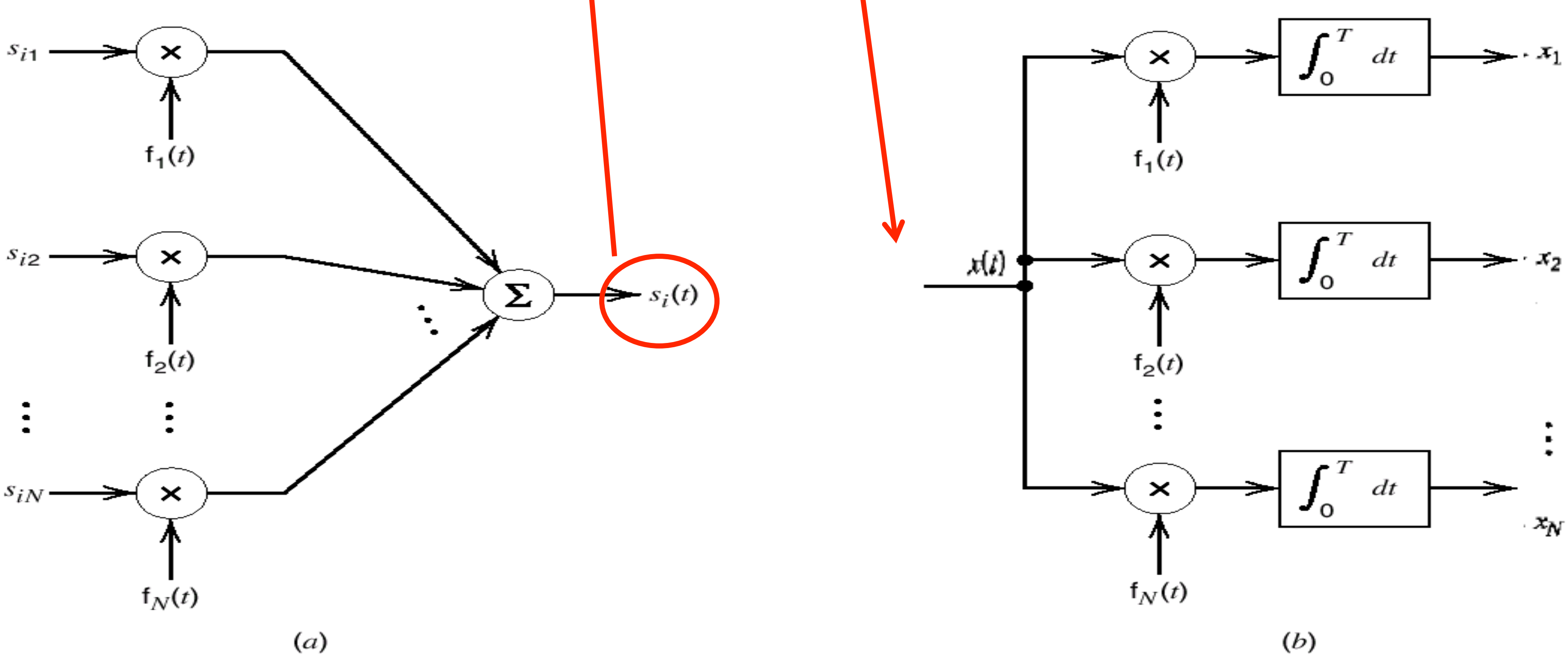


Figure 5.3 ***(a) Synthesizer for generating the signal $s_i(t)$.***
(b) Analyzer for generating the set of signal vectors $\{s_i\}$.

5.3 Conversion of the Continuous AWGN Channel into a Vector Channel

Suppose that the input to the bank of N product integrators or correlators in Figure 5.3b is not the transmitted signal $s_i(t)$ but rather the received signal $x(t)$ defined in accordance with the idealized AWGN channel of Figure 5.2. That is to say,

$$x(t) = s_i(t) + w(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \tag{5.28}$$

where $w(t)$ is a sample function of a white Gaussian noise process $W(t)$ of zero mean and power spectral density $N_0/2$. Correspondingly, we find that the output of correlator j, say, is the sample value of a random variable X_j, as shown by

$$\begin{aligned} x_j &= \int_0^T x(t)\phi_j(t)dt \\ &= s_{ij} + w_j, \qquad j = 1, 2, \ldots, N \end{aligned} \tag{5.29}$$

The first component, s_{ij}, is a deterministic quantity contributed by the transmitted signal $s_i(t)$; it is defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \tag{5.30}$$

The second component, w_j, is the sample value of a random variable W_j that arises because of the presence of the channel noise $w(t)$; it is defined by

$$w_j = \int_0^T w(t)\phi_j(t)dt \tag{5.31}$$

STATISTICAL CHARACTERIZATION OF THE CORRELATOR OUTPUTS

Let x_j denote the random variable of the output of correlator j.

The mean of X_j for signal $s_i(t)$ is (from 5.29)

$$\mu_{x_j} = E[X_j]$$
$$= E[s_{ij}] + E[W_j]$$
$$= s_{ij} \qquad (5.35)$$

$$\sigma^2_{x_j} = E[(X_j - s_{ij})^2]$$
$$= E[W_j^2] \qquad (5.36)$$

According to (5.31)

$$W_j = \int_0^T W(t)\phi_j(t)dt$$
$$\sigma^2_{x_j} = E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_j(u)du\right]$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)E[W(t)W(u)]dtdu$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)R_w(t,u)dtdu$$
$$= \frac{No}{2}\int_0^T \phi_j(t)dt = \frac{No}{2} \qquad (5.41)$$

Because $R_w(t,u) = \dfrac{No}{2}\delta(t-u)$ (5.39)

$$\begin{aligned}\operatorname{cov}[X_j X_k] &= E[(X_j-\mu_{x_j})(X_k-\mu_{x_k})]\\ &= E[(X_j-s_{ij})(X_k-s_{ik})]\\ &= E[W_j W_k]\\ &= \int_0^T\int_0^T \phi_j(t)\phi_k(u)R_w(t,u)\,dt\,du\\ &= \frac{N_0}{2}\int_0^T \phi_j(t)\phi_k(u)R_w(t,u)\,dt\,du\\ &= 0 \quad , \quad j\neq k\end{aligned}$$

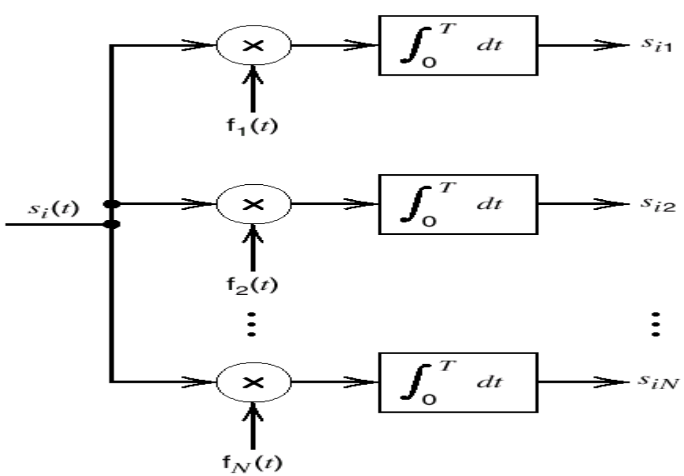
The X_j are mutually uncorrelated.

Since the X_j are Gaussian, they are statscally independent.

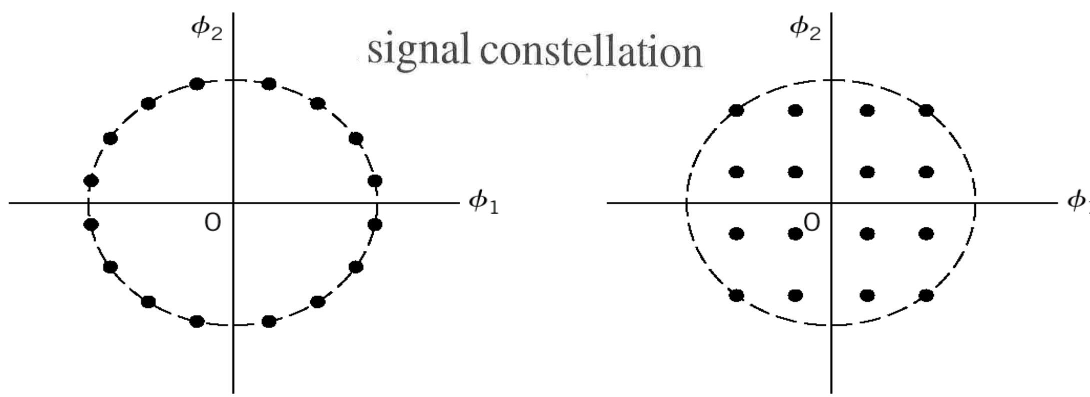
i.e the sampled correlator outputs are independent Gaussian random variables.

5.5 Coherent Detection of Signals in Noise: Maximum Likelihood Decoding

Suppose that in each time slot of duration T seconds, one of the M possible signals $s_1(t)$, $s_2(t), \ldots, s_M(t)$ is transmitted with equal probability, $1/M$. For geometric signal representation, the signal $s_i(t), i = 1, 2, \ldots, M$, is applied to a bank of correlators, with a common input and supplied with an appropriate set of N orthonormal basis functions. The resulting correlator outputs define the *signal vector* $\mathbf{s}_i$. Since knowledge of the signal vector $\mathbf{s}_i$ is as good as knowing the transmitted signal $s_i(t)$ itself, and vice versa, we may represent $s_i(t)$ by a point in a Euclidean space of dimension $N \leq M$. We refer to this point as the *transmitted signal point* or *message point*. The set of message points corresponding to the set of transmitted signals $\{s_i(t)\}_{i=1}^{M}$ is called a *signal constellation*.



(b)



(a) (b)

Signal constellation for (a) M-ary PSK and (b) corresponding M-ary QAM, for M = 16.

However, the representation of the received signal $x(t)$ is complicated by the presence of additive noise $w(t)$. We note that when the received signal $x(t)$ is applied to the bank of N correlators, the correlator outputs define the observation vector **x**. From Equation (5.48), the vector **x** differs from the signal vector $\mathbf{s}_i$ by the *noise vector* **w** whose orientation is completely random. The noise vector **w** is completely characterized by the noise $w(t)$;

Now, based on the observation vector **x**, we may represent the received signal $x(t)$ by a point in the same Euclidean space used to represent the transmitted signal. We refer to this second point as the *received signal point.* The received signal point wanders about the message point in a completely random fashion, in the sense that it may lie anywhere inside a Gaussian-distributed "cloud" centered on the message point. This is illustrated in Figure 5.7*a* for the case of a three-dimensional signal space. For a particular realization of the noise vector **w** (i.e., a particular point inside the random cloud of Figure 5.7*a*), the relationship between the observation vector **x** and the signal vector $\mathbf{s}_i$ is as illustrated in Figure 5.7*b*.

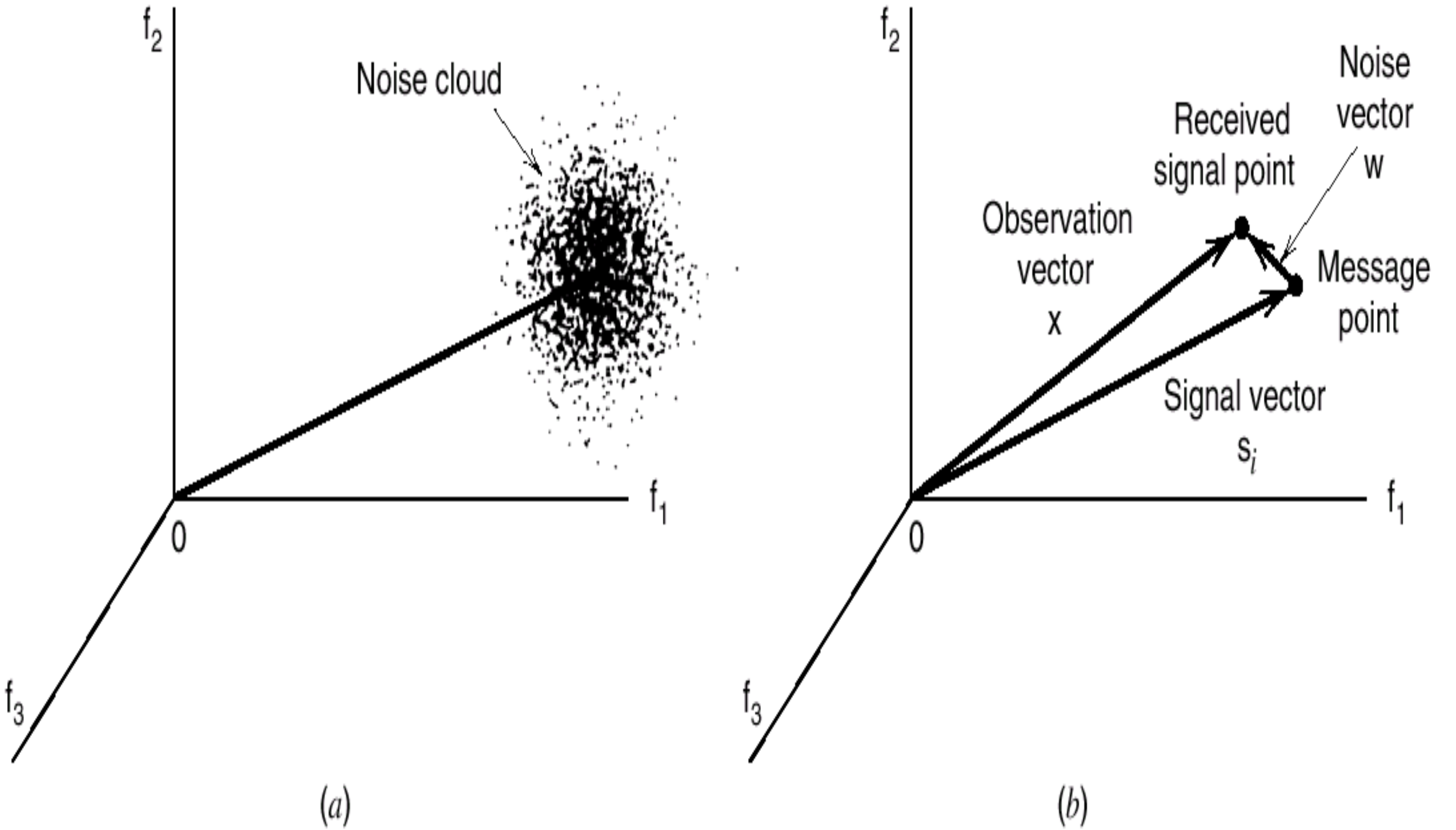


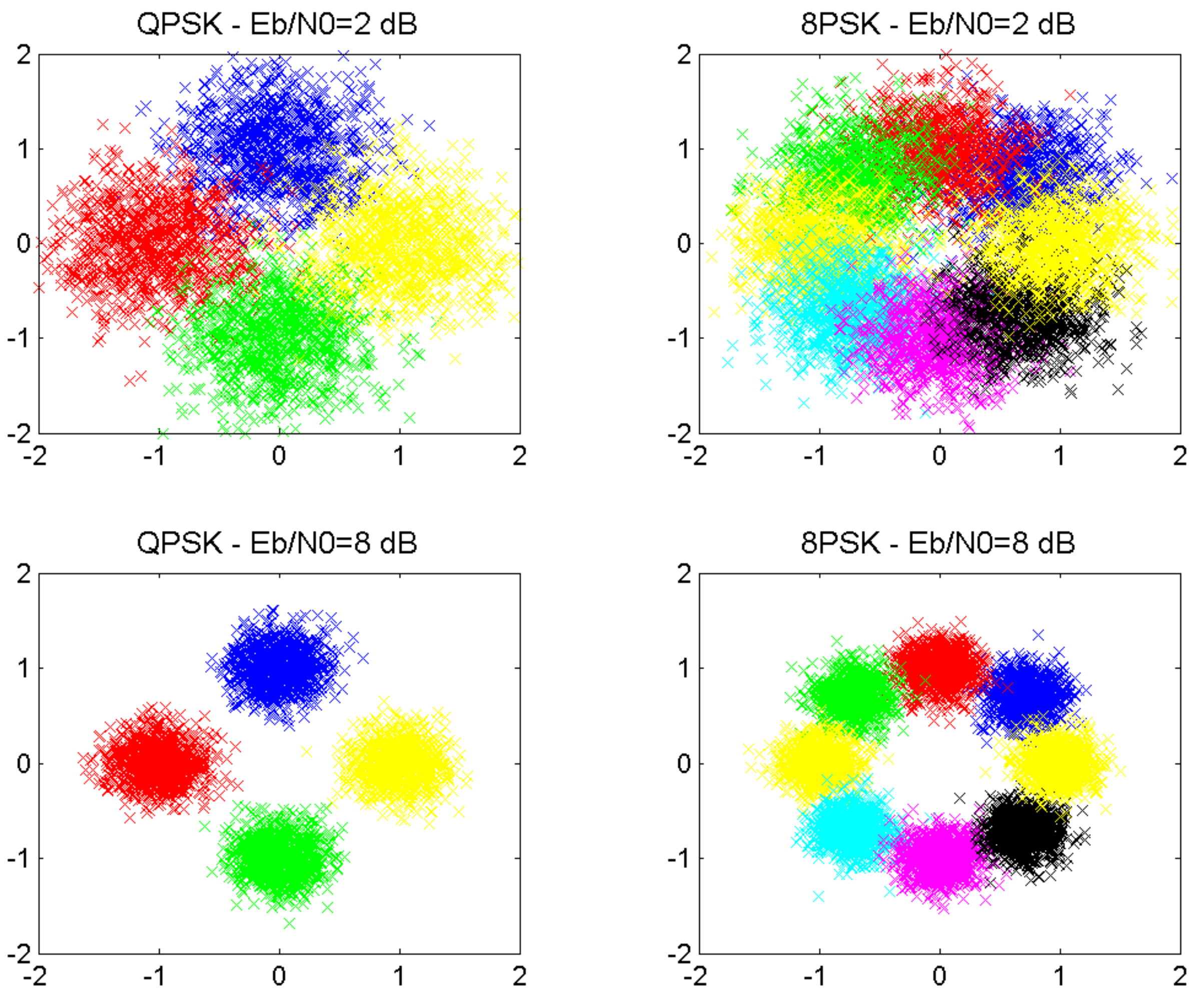
Figure 5.7 ***Illustrating the effect of noise perturbation, depicted in (a), on the location of the received signal point, depicted in (b).***

AWGN is

equivalent to an N-dimensional *vector channel* described by the observation vector

$$\mathbf{x} = \mathbf{s}_i + \mathbf{w}, \qquad i = 1, 2, \ldots, M \tag{5.48}$$

Example of samples of matched filter output for some bandpass modulation schemes



Suppose that, given the observation vector $\mathbf{x}$, we make the decision $\hat{m} = m_i$. The probability of error in this decision, which we denote by $P_e(m_i|\mathbf{x})$, is simply

$$\begin{aligned} P_e(m_i|\mathbf{x}) &= P(m_i \text{ not sent}|\mathbf{x}) \\ &= 1 - P(m_i \text{ sent}|\mathbf{x}) \end{aligned} \tag{5.52}$$

The decision-making criterion is to minimize the probability of error in mapping each given observation vector $\mathbf{x}$ into a decision. On the basis of Equation (5.52), we may therefore state the *optimum decision rule*:

$$\begin{gathered} \text{Set } \hat{m} = m_i \text{ if} \\ P(m_i \text{ sent}|\mathbf{x}) \geq P(m_k \text{ sent}|\mathbf{x}) \qquad \text{for all } k \neq i \end{gathered} \tag{5.53}$$

where $k = 1, 2, \ldots, M$. This decision rule is referred to as the *maximum a posteriori probability (MAP) rule.*

The condition of Equation (5.53) may be expressed more explicitly in terms of the *a priori* probabilities of the transmitted signals and in terms of the likelihood functions. Using Bayes' rule in Equation (5.53), and for the moment ignoring possible ties in the decision-making process, we may restate the MAP rule as follows:

$$l(m_k) = \sum_{j=1}^{N} (x_j - s_{kj})^2$$

$$\text{Set } \hat{m} = m_i \text{ if}$$

$$\frac{p_k f_{\mathbf{X}}(\mathbf{x}|m_k)}{f_{\mathbf{X}}(\mathbf{x})} \text{ is maximum for } k = i \tag{5.54}$$

where p_k is the *a priori* probability of transmitting symbol m_k, $f_{\mathbf{X}}(\mathbf{x}|m_k)$ is the conditional probability density function of the random observation vector $\mathbf{X}$ given the transmission of symbol m_k, and $f_{\mathbf{X}}(\mathbf{x})$ is the unconditional probability density function of $\mathbf{X}$. In Equation (5.54) we may note the following:

- The denominator term $f_{\mathbf{X}}(\mathbf{x})$ is independent of the transmitted symbol.
- The *a priori* probability $p_k = p_i$ when all the source symbols are transmitted with equal probability.
- The conditional probability density function $f_{\mathbf{X}}(\mathbf{x}|m_k)$ bears a one-to-one relationship to the log-likelihood function $l(m_k)$. $= \sum_{j=1}^{N} (x_j - s_{kj})^2$

This decision rule is referred to as the *maximum likelihood rule*, and the device for its implementation is correspondingly referred to as the *maximum likelihood decoder*. According to Equation (5.55), a maximum likelihood decoder computes the log-likelihood functions as metrics for all the M possible message symbols, compares them, and then decides in favor of the maximum. Thus the maximum likelihood decoder differs from the maximum *a posteriori* decoder in that it assumes equally likely message symbols.

graphical interpretation of the maximum likelihood decision

rule. Let Z denote the N-dimensional space of all possible observation vectors $\mathbf{x}$. We refer to this space as the *observation space*. Because we have assumed that the decision rule must say $\hat{m} = m_i$, where $i = 1, 2, \ldots, M$, the total observation space Z is correspondingly partitioned into *M-decision regions*, denoted by $Z_1, Z_2, \ldots, Z_M$. Accordingly, we may restate the decision rule of Equation (5.55) as follows:

$$\begin{gathered}\text{Observation vector } \mathbf{x} \text{ lies in region } Z_i \text{ if} \\ \text{the Euclidean distance } \|\mathbf{x} - \mathbf{s}_k\| \text{ is minimum for } k = i\end{gathered} \tag{5.59}$$

Equation (5.59) states that *the maximum likelihood decision rule is simply to choose the message point closest to the received signal point*, which is intuitively satisfying.

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 = \|\mathbf{x} - \mathbf{s}_k\|^2$$

$\| \mathbf{x} - \mathbf{s}_k \|$ is the Euclidean distance

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 = \| \mathbf{x} - \mathbf{s}_k \|^2$$

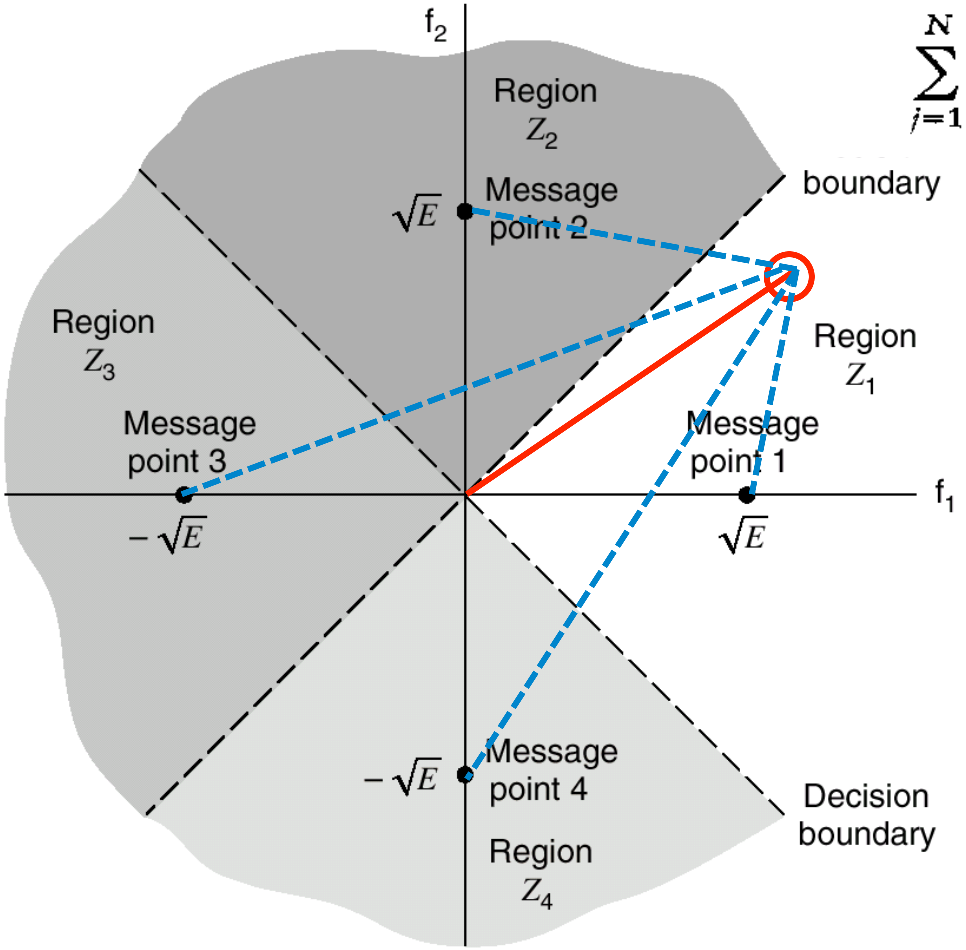


Figure 5.8 Illustrating the partitioning of the observation space into decision regions for the case when $N = 2$ and $M = 4$; it is assumed that the M transmitted symbols are equally likely.

$M = 4$ signals and $N = 2$ dimensions, assuming that the signals are transmitted with equal energy, E, and equal probability.

In practice, the need for squarers in the decision rule of Equation (5.59) is avoided by recognizing that

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 = \sum_{j=1}^{N} x_j^2 - 2 \sum_{j=1}^{N} x_j s_{kj} + \sum_{j=1}^{N} s_{kj}^2 \tag{5.60}$$

The first summation term of this expansion is independent of the index k and may therefore be ignored. The second summation term is the inner product of the observation vector $\mathbf{x}$ and signal vector $\mathbf{s}_k$. The third summation term is the energy of the transmitted signal $s_k(t)$. Accordingly, we may formulate a decision rule equivalent to that of Equation (5.59) as follows:

Observation vector $\mathbf{x}$ lies in region Z_i if

$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2} E_k \text{ is maximum for } k = i \tag{5.61}$$

where E_k is the energy of the transmitted signal $s_k(t)$:

$$E_k = \sum_{j=1}^{N} s_{kj}^2 \tag{5.62}$$

From Equation (5.61) we deduce that, for an AWGN channel, the decision regions are regions of the N-dimensional observation space Z, bounded by linear [$(N - 1)$-dimensional hyperplane] boundaries. Figure 5.8 shows the example of decision regions for

Summary of Coherent Detection of signals in Noise ML Decoding

(1) Analytically

(2) Graphically

Summary of Coherent Detection of signals in Noise ML Decoding (1)

Detection problem:

Given x, perform a mapping from x to an estimate $\hat{m}_i$ of m_i, in a way that would minimize the probability of error.

The prob. of error denoted by $P_e(m_i|\mathrm{x})$ is

$$P_e(m_i|\mathrm{x}) = P(m_i \text{ not sent } |\mathrm{x}) = 1 - P(m_i \text{ sent } |\mathrm{x}) \quad (5.52)$$

The optimum decsion rule is

$$\text{set } \hat{m} = m_i \text{ if } P(m_i \text{ sent } |\mathrm{x}) \geq P(m_k \text{ sent } |\mathrm{x}) \text{ for all } k \neq i \quad (5.53)$$

which is also called the maximum a posteriori probability (MAP) rule.

In terms to the a priori prob. of $\{m_i\}$, using Bayes' rule, we may restate the MAP rule as

$$\text{set } \hat{m} = m_i \text{ if } \frac{P_k f_{\mathrm{x}}(\mathrm{x}|m_k)}{f_{\mathrm{x}}(\mathrm{x})} \text{ is maximum for } k = i \quad (5.54)$$

$\{(5.53) \Rightarrow P(m_k \text{ sent } |\mathrm{X}) \text{ is maximum for } i = k\}$

where p_k is the a priori prob. of m_k

Note that

1. $f_{\mathrm{x}}(\mathrm{x})$ is indep. of $\{m_i\}$
2. If $\{m_i\}$ are equally likely, $p_k = p_i = p$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' rule.

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Summary of Coherent Detection of signals in Noise ML Decoding-Graphically (2)

(5.55) is called as the maximum likelihood rule.

The maximum likelihood decoder differs from the maximum a posteriori decoder (Assum. of p_k = constant)

Let Z denote the N - dim space (observation space).

We may partition Z into M - decision regions denoted by $Z_1, Z_2, \ldots, Z_M$

Observation vector x lies in Z_i if

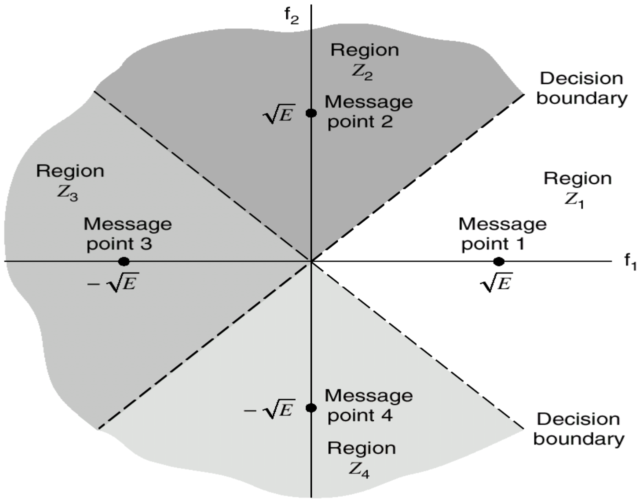
$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \|\mathbf{x} - s_k\|^2 \text{ is min. for } k = i \quad (5.57)$$

$$\Longrightarrow \mathbf{x} \in Z_i \text{ , if } \|\mathbf{x} - s_k\| \text{ is min. for } k = i \quad (5.59)$$

$\Longrightarrow$ to choose the message point closest to the received signal point.

$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \sum_{j=1}^{N} x_j^2 - 2\sum_{j=1}^{N} x_j s_{kj} + \sum_{j=1}^{N} s_{kj}^2 \quad (5.60)$$

indep. of k (first term); energy of $s_k(t) = E_k$ (last term)



Equivalently we have

$\mathbf{x} \in Z_i$ if

$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2}E_k \text{ is max. for } k = i$$

Week (6)- Lecture (2) Signal-Space Analysis

5.6 *Correlation Receiver*

5.7 *Probability of Error*

5.6 *Correlation Receiver*

$\{s_i(t)\}$ are equally likely.

The correlation receiver consists of two parts :

* detector (correlator)
* decoder

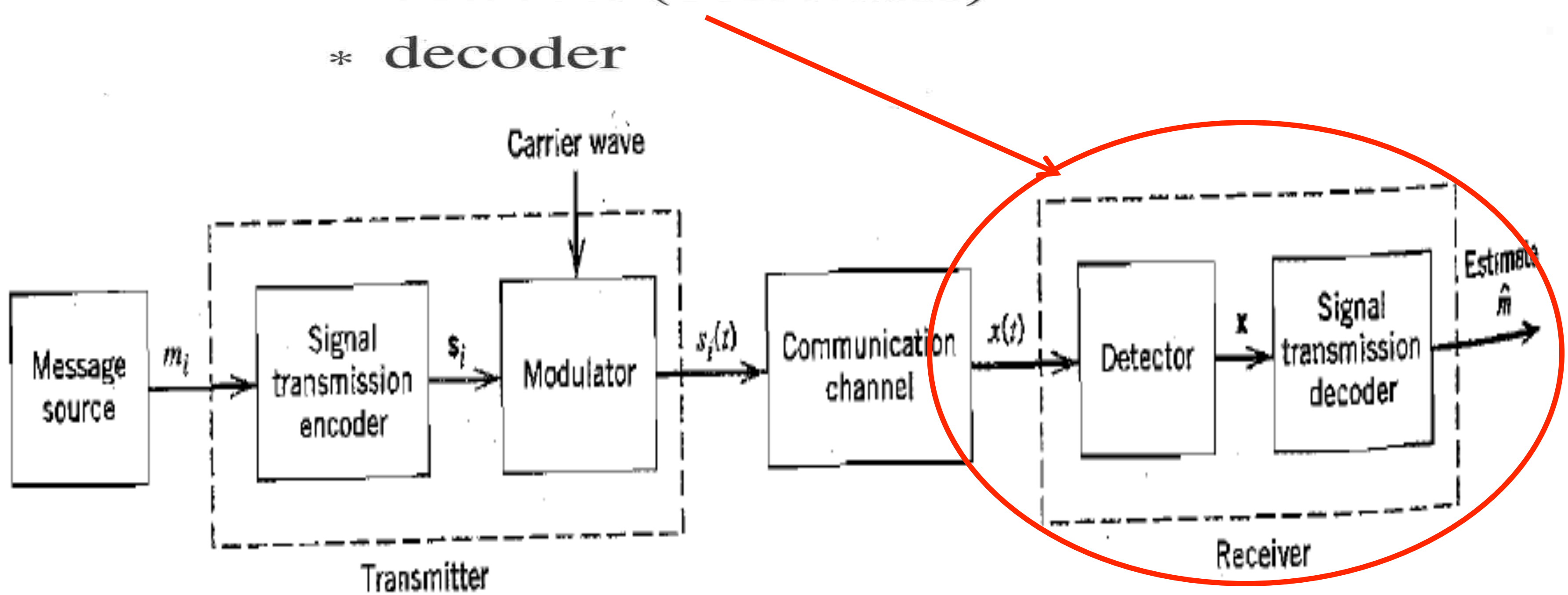


FIGURE 6.2 Functional model of passband data transmission system.

The optimum receiver of Figure 5.9 is commonly referred to as a *correlation receiver.*

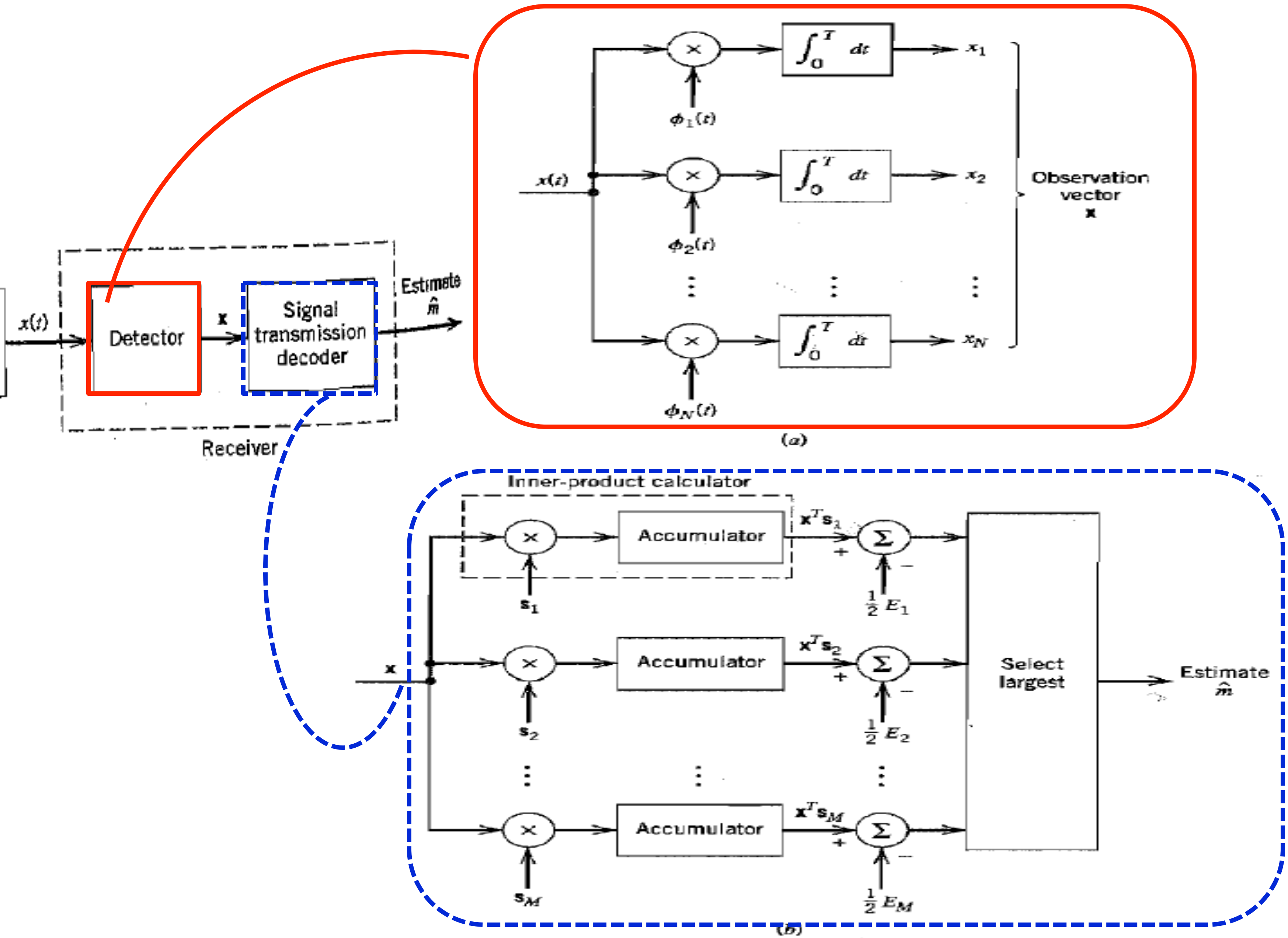
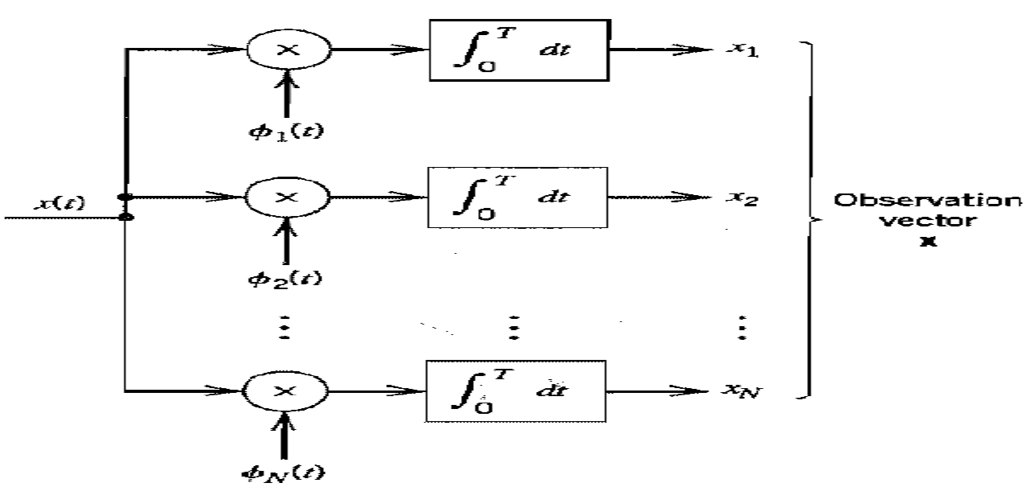


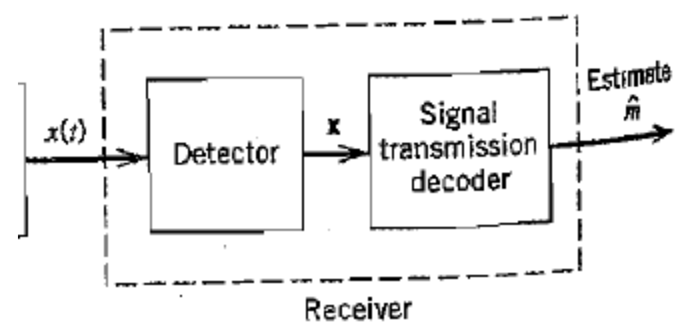
FIGURE 5.9 (*a*) Detector or demodulator. (*b*) Signal transmission decoder.

5.6 *Correlation Receiver*

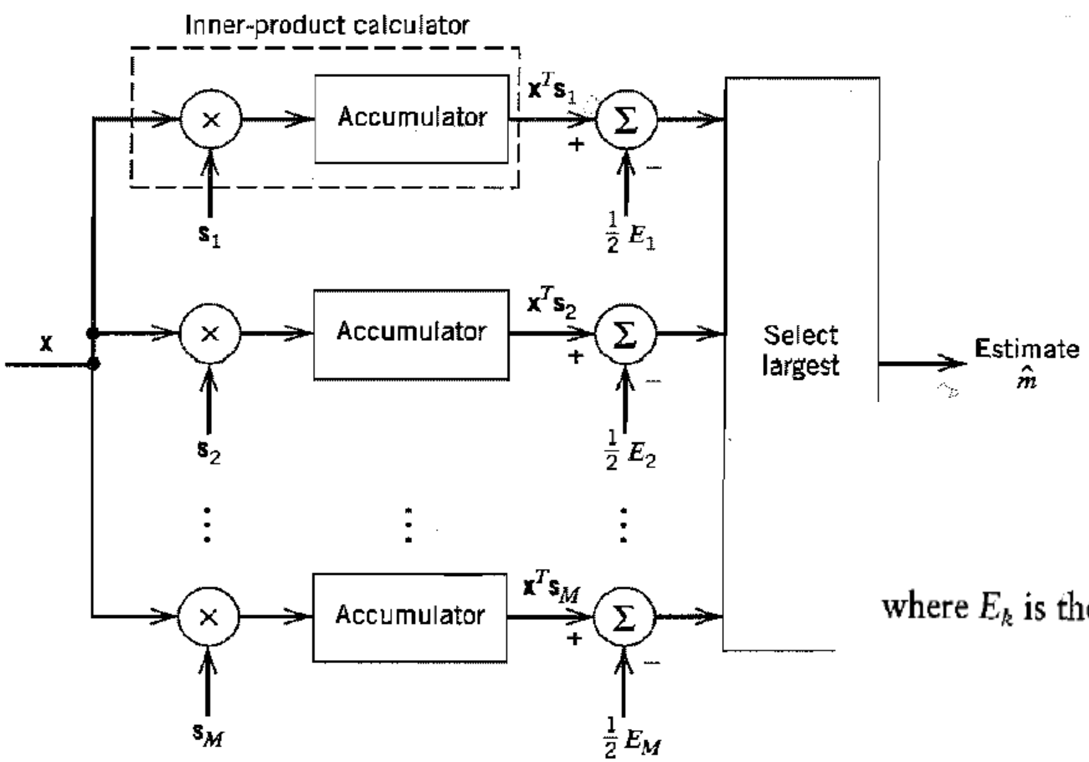
From the material presented in the previous sections, we find that for an AWGN channel and for the case when the transmitted signals $s_1(t), s_2, \ldots, s_M(t)$ are equally likely, the optimum receiver consists of two subsystems, which are detailed in Figure 5.9 and described here:

1. The *detector* part of the receiver is shown in Figure 5.9*a*. It consists of a bank of M *product-integrators* or *correlators*, supplied with a corresponding set of coherent reference signals or orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ that are generated locally. This bank of correlators operates on the received signal $x(t)$, $0 \leq t \leq T$, to produce the observation vector **x**.





2. The second part of the receiver, namely, the *signal transmission decoder* is shown in Figure 5.9*b*. It is implemented in the form of a maximum-likelihood decoder that operates on the observation vector $\mathbf{x}$ to produce an estimate, $\hat{m}$, of the transmitted symbol m_i, $i = 1, 2, \ldots, M$, in a way that would minimize the average probability of symbol error. In accordance with Equation (5.61), the N elements of the observation vector $\mathbf{x}$ are first multiplied by the corresponding N elements of each of the M signal vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$, and the resulting products are successively summed in accumulators to form the corresponding set of inner products $\{\mathbf{x}^T\mathbf{s}_k \mid k = 1, 2, \ldots, M\}$. Next, the inner products are corrected for the fact that the transmitted signal energies may be unequal. Finally, the largest in the resulting set of numbers is selected and an appropriate decision on the transmitted message is made.



Observation vector $\mathbf{x}$ lies in region Z_i if the Euclidean distance $\|\mathbf{x} - \mathbf{s}_k\|$ is minimum for $k = i$

Observation vector $\mathbf{x}$ lies in region Z_i if

$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2} E_k \text{ is maximum for } k = i$$

where E_k is the energy of the transmitted signal $s_k(t)$:

$$E_k = \sum_{j=1}^{N} s_{kj}^2$$

▶ EXAMPLE 4.1 Matched Filter for Rectangular Pulse

Consider a signal $g(t)$ in the form of a rectangular pulse of amplitude A and duration T, as shown in Figure 4.2*a*. In this example, the impulse response $h(t)$ of the matched filter has exactly the same waveform as the signal itself. The output signal $g_o(t)$ of the matched filter produced in response to the input signal $g(t)$ has a triangular waveform, as shown in Figure 4.2*b*.

The maximum value of the output signal $g_o(t)$ is equal to kA^2T, which is the energy of the input signal $g(t)$ scaled by the factor k; this maximum value occurs at $t = T$, as indicated in Figure 4.2*b*.

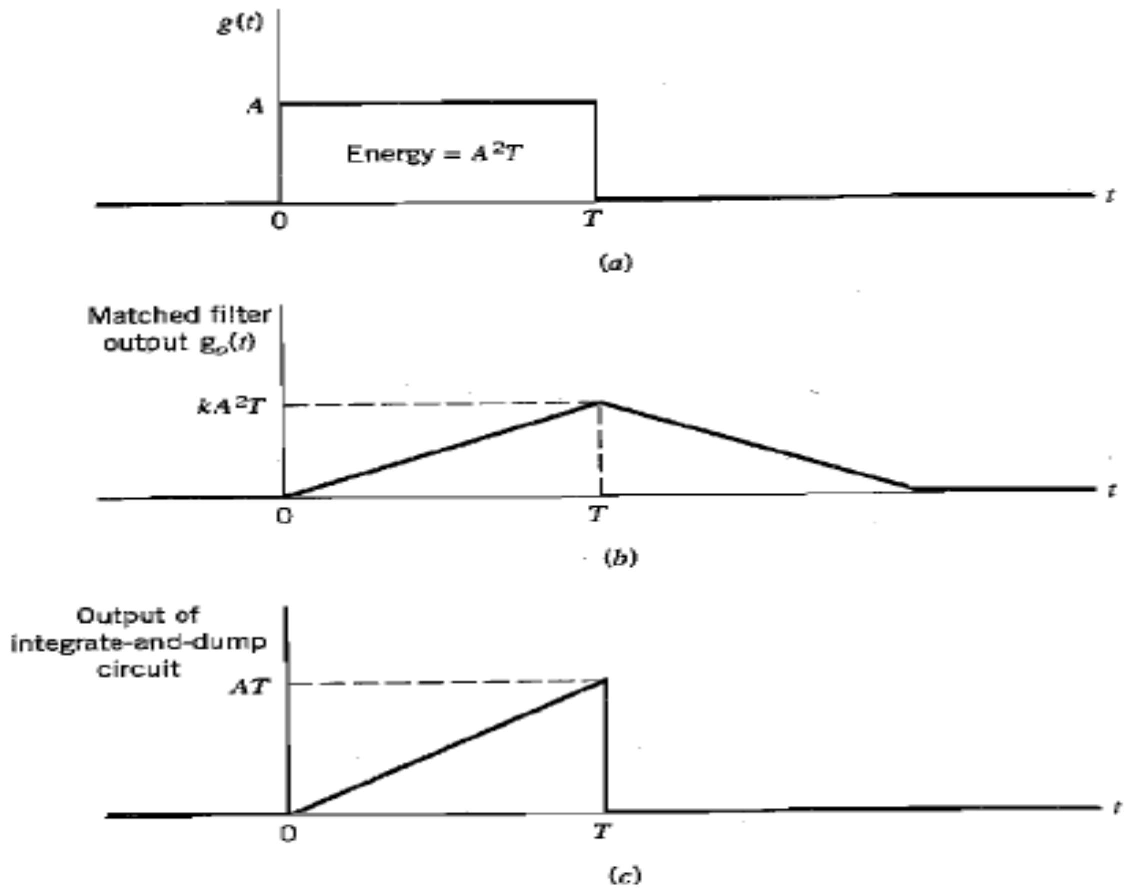


FIGURE 4.2 (*a*) Rectangular pulse. (*b*) Matched filter output. (*c*) Integrator output.

Equivalence of Correlation and Matched Filter Receivers

Recall the output of the matched filter

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau) h_j(t-\tau) d\tau \qquad (5.63)$$

let $h_j(t) = \phi_j(T-\tau)$ (5.64)

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau) \phi_j(T-t+\tau) d\tau \qquad (5.65)$$

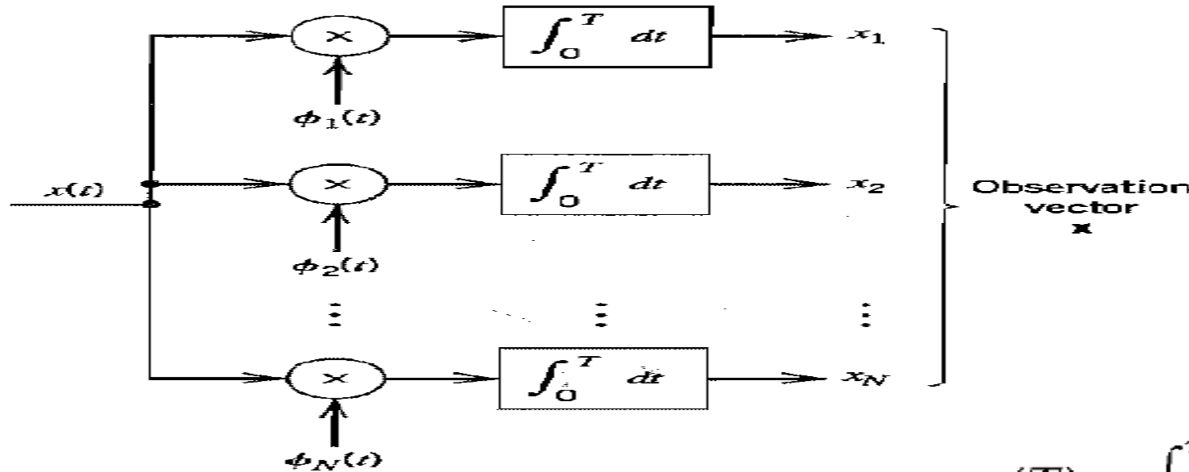
sample at $t = T$, and $\phi_j(t) = 0$, $t \langle 0$, or $t \rangle T$

$$y_j(T) = \int_{-\infty}^{\infty} x(\tau) \phi_j(\tau) d\tau$$

$$= \int_{0}^{T} x(\tau) \phi_j(\tau) d\tau \qquad (5.66)$$

↑

correlator



$$y_j(T) = \int_0^T x(\tau)\phi_j(\tau)d\tau$$

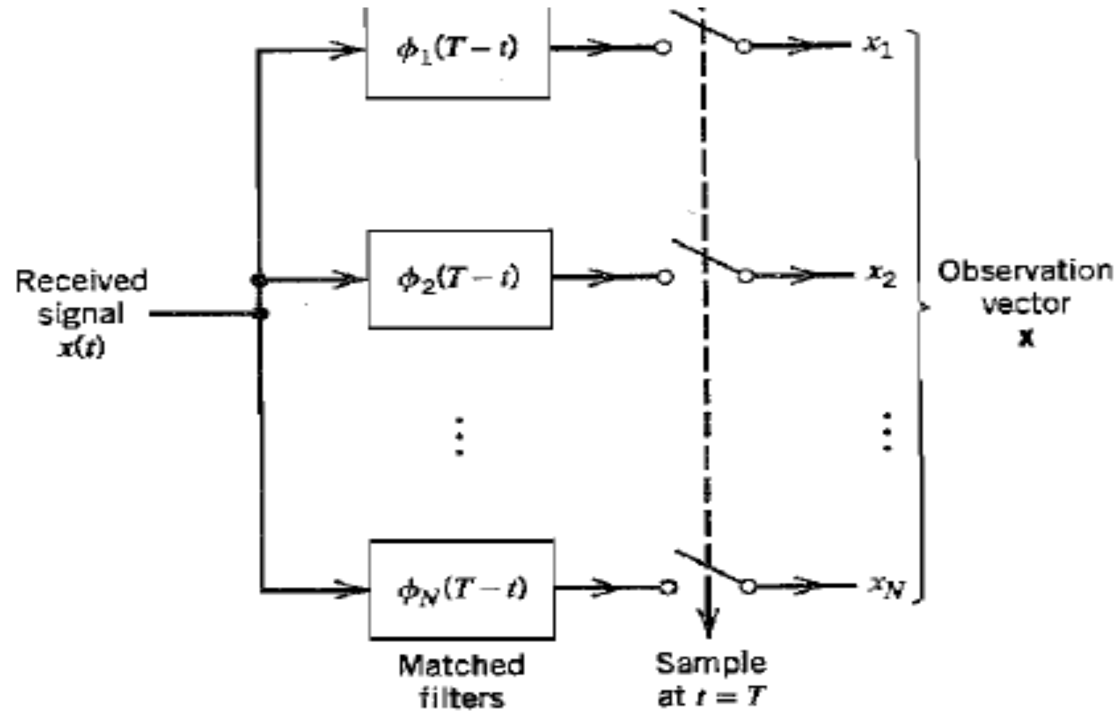


FIGURE 5.10 Detector part of matched filter receiver; the signal transmission decoder is as shown in Fig. 5.9*b*.

5.7 Probability of Error

readily see that the *average probability of symbol error*, P_e is

$$
\begin{aligned}
P_e &= \sum_{i=1}^{M} p_i \, P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\
&= \frac{1}{M} \sum_{i=1}^{M} P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\
&= 1 - \frac{1}{M} \sum_{i=1}^{M} P(\mathbf{x} \text{ lies in } Z_i | m_i \text{ sent})
\end{aligned}
\qquad (5.67)
$$

Pc

where we have used standard notation to denote the probability of an event and the conditional probability of an event. Since $\mathbf{x}$ is the sample value of random vector $\mathbf{X}$, we may rewrite Equation (5.67) in terms of the likelihood function (when m_i is sent) as follows:

$$
P_e = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i} f_{\mathbf{X}}(\mathbf{x} | m_i) \, d\mathbf{x} \qquad (5.68)
$$

summary

5.7 Probability of Error

The average prob. of symbol error

$$P_e = \sum_{i=1}^{M} p_i P(\text{x does not lie in } Z_i \,|\, m_i \text{ sent})$$

$$= \frac{1}{M} \sum_{i=1}^{M} P(\text{x does not lie in } Z_i \,|\, m_i \text{ sent}) \quad (5.67)$$

$$= 1 - \frac{1}{M} \sum_{i=1}^{M} P(\text{x lies in } Z_i \,|\, m_i \text{ sent}) \quad (5.68)$$

Invariance of P_e to Rotation and Translation

Changes of coordinates does not affect P_e

1. P_e depends on $\|\mathrm{x} - s_k\|$

If a signal constellation is rotated by an orthonormal transformation, that is ,

$$\mathrm{s}_{\mathrm{i,rotate}} = \mathrm{Q}\mathrm{s}_i\,,\ \ i = 1,2,\ldots,M$$

where Q is an orthonormal matrix, then the probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.

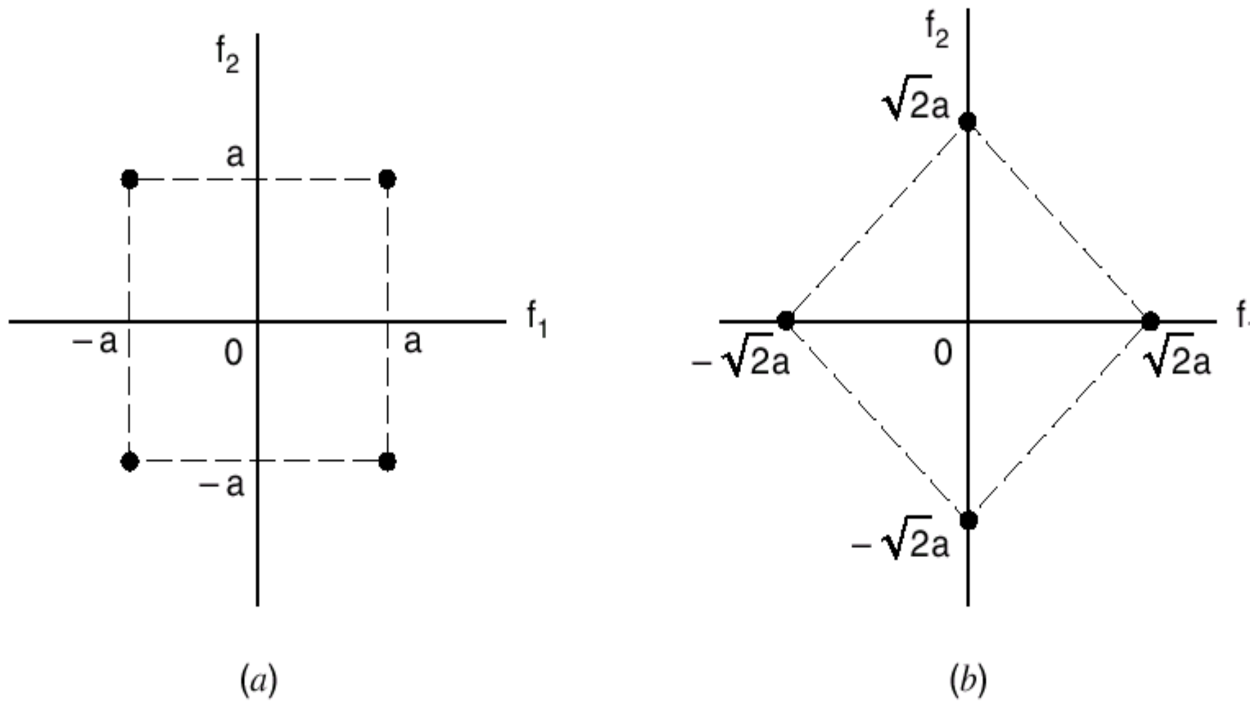


Figure 5.11 ***A pair of signal constellations for illustrating the principle of rotational invariance.***

For Translation

$$\mathbf{s}_{i,\text{translate}} = \mathbf{s}_i - \mathbf{a}, \; i = 1,2,\ldots,M \qquad (5.77)$$

$$\mathbf{x}_{\text{translate}} = \mathbf{x} - \mathbf{a} \qquad (5.78)$$

$$\begin{aligned} &\left\| \mathbf{x}_{\text{translate}} - \mathbf{s}_{i,\text{translate}} \right\| \\ &= \left\| \mathbf{x} - \mathbf{a} - \mathbf{s}_i + \mathbf{a} \right\| \\ &= \left\| \mathbf{x} - \mathbf{s}_i \right\| \qquad \text{for all i} \end{aligned} \qquad (5.79)$$

The priciple of translational invariance:

If a signal constellation is translate by a constant vector amount, then the probility of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.

Minimum Energy Signals

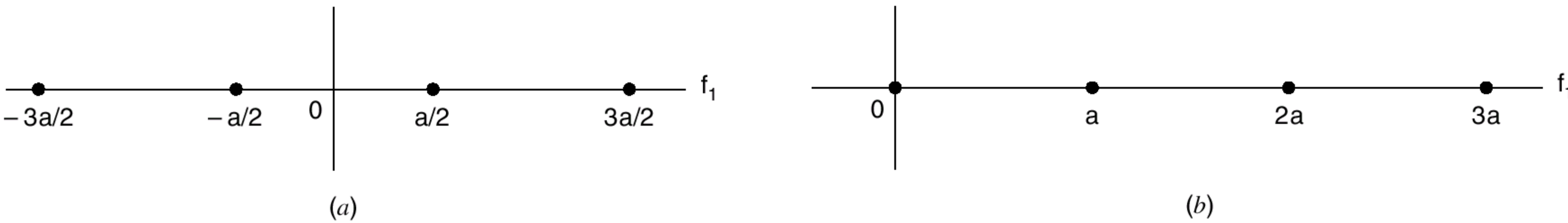


Figure 5.12 ***A pair of signal constellations for illustrating the principle of translational invariance.***

consider $\{m_i\}$ represented by $\{s_i\}$

The average energy of $\{s_i\}$ translated by a

$$\xi_{\text{translate}} = \sum_{i=1}^{M} \| s_i - a \|^2 p_i \qquad (5.80)$$

p_i : the prob. of m_i

$$\| s_i - a \|^2 = \| s_i \|^2 - 2a^T s_i + \| a \|^2$$

$$\xi_{\text{translate}} = \sum_{i=1}^{M} \left\| s_i \right\|^2 p_i - 2\sum_{i=1}^{M} a^T s_i p_i + \left\| a \right\|^2 \sum_{i=1}^{M} p_i$$

$$= \xi - 2a^T E[s] + \left\| a \right\|^2 \qquad (5.81)$$

↑ The energy of original constellation (ξ)

$$E[s] = \sum_{i=1}^{M} s_i p_i \qquad (5.82)$$

$$\frac{\partial \xi_{\text{translate}}}{\partial a} = 0 \Rightarrow a_{\min} = E[s] \qquad (5.83)$$

$$\xi_{\text{trnaslate,min}} = \xi - \left\| a_{\min} \right\|$$

The minimum energy translate :

Given a signal costellation $\{s_i\}_{i=1}^{M}$, the corresponding signal constellation with minimum average energy is obtained by subtracting from each siganl vector s_i in the given constellation an amount equal to the constant vector $E[s]$, where $E[s]$ is defined by Equation (5.82)

1. If the constellation is circularly symmetric, $P_e(m_i)$ is the same for all i (e.q. M - ary PSK)

$$P_e \le \frac{1}{2}\sum_{\substack{k=1 \\ k\neq i}}^{M} erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \quad , \quad \text{for all } i \qquad (5.92)$$

2. Define $d_{\min}$ = minimum d_{ik} (5.93)

$$erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \le erfc\left(\frac{d_{\min}}{2\sqrt{N_0}}\right)$$

$$P_e \le \frac{(M-1)}{2} erfc\left(\frac{d_{\min}}{2\sqrt{N_0}}\right)$$

Q Function & erfc

3. The *error function* denoted by erf(u), is defined in a number of different ways in the literature. We shall use the following definition:

$$\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2)\, dz$$

The error function has two useful properties:

(i) $\operatorname{erf}(-u) = -\operatorname{erf}(u)$
This is known as the *symmetry relation.*

(ii) As u approaches infinity, erf(u) approaches unity; that is,

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-z^2)\, dz = 1$$

The *complementary error function* is defined by

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)\, dz$$

which is related to the error function as follows:

$$\operatorname{erfc}(u) = 1 - \operatorname{erf}(u)$$

The *complementary error function* is defined by

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)\, dz$$

erfc $= \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)\, dz$

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty \exp\left(-\frac{x^2}{2}\right) dx$$

The Q-function defines the area under the standardized Gaussian tail. The Q-function is related to the complementary error function as

$z \rightarrow \frac{x}{\sqrt{2}}$

erfc $= \frac{2}{\sqrt{2}\sqrt{\pi}} \int_u^\infty \exp\left(-\frac{x^2}{2}\right) dx$

$$Q(v) = \frac{1}{2} \operatorname{erfc}\left(\frac{v}{\sqrt{2}}\right)$$

$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$

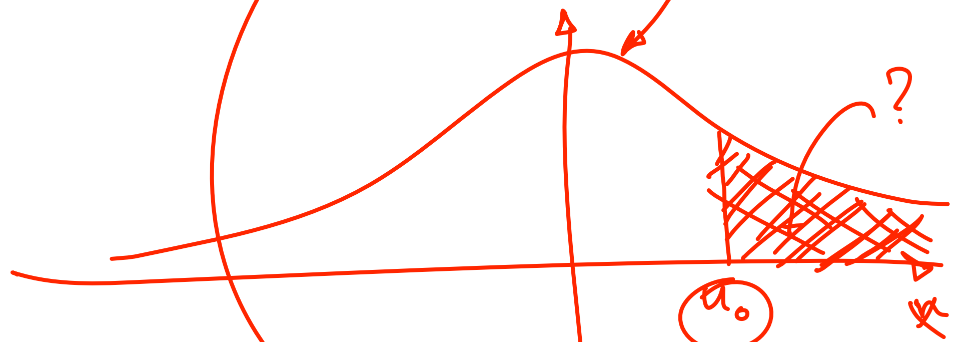
Conversely, putting $u = v/\sqrt{2}$, we have

$$\operatorname{erfc}(u) = 2Q(\sqrt{2}u)$$

$$Q = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{u}^{\infty} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \, dx$$

$$z = \frac{x}{\sigma}$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{u_0}^{\infty} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \, d$$

$$Q\left(\frac{u_0}{\sigma}\right)$$