M ary - PSK

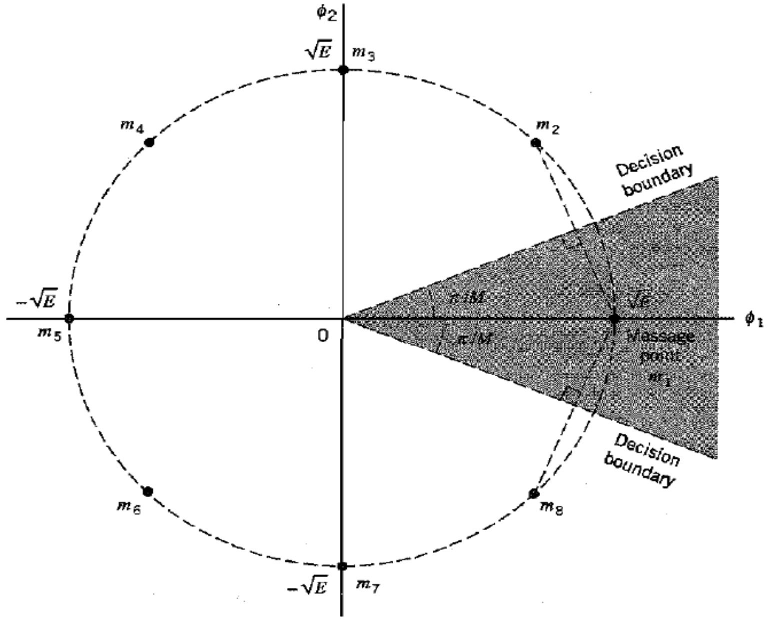
M-ARY PSK

QPSK is a special case of *M-ary PSK*, where the phase of the carrier takes on one of M possible values, namely, $\theta_i = 2(i-1)\pi/M$, where $i = 1, 2, \ldots, M$. Accordingly, during each signaling interval of duration T, one of the M possible signals

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \qquad i - 1, 2, \ldots, M \tag{6.46}$$

is sent, where E is the signal energy per symbol. The carrier frequency $f_c = n_c/T$ for some fixed integer n_c.

Each $s_i(t)$ may be expanded in terms of the same two basis functions $\phi_1(t)$ and $\phi_2(t)$ defined in Equations (6.25) and (6.26), respectively. The signal constellation of M-ary PSK is therefore two-dimensional. The M message points are equally spaced on a circle of radius $\sqrt{E}$ and center at the origin, as illustrated in Figure 6.15*a*, for the case of *octaphase-shift-keying* (i.e., $M = 8$).



(*b*)

FIGURE 6.15 (*a*) Signal-space diagram for octaphase-shift keying (i.e., $M = 8$). The decision boundaries are shown as dashed lines. (*b*) Signal-space diagram illustrating the application of the

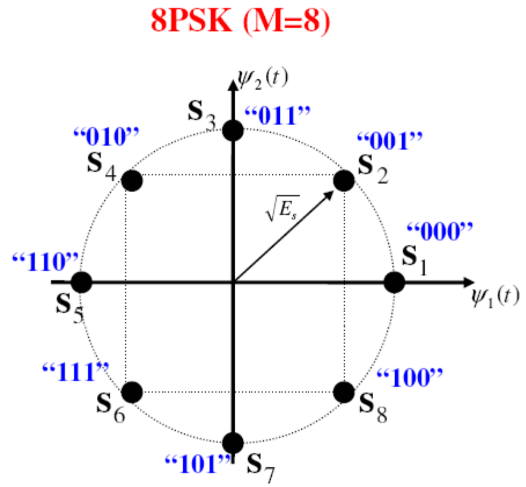
$$d_{12} = d_{18} = 2\sqrt{E}\sin\left(\frac{\pi}{M}\right)$$

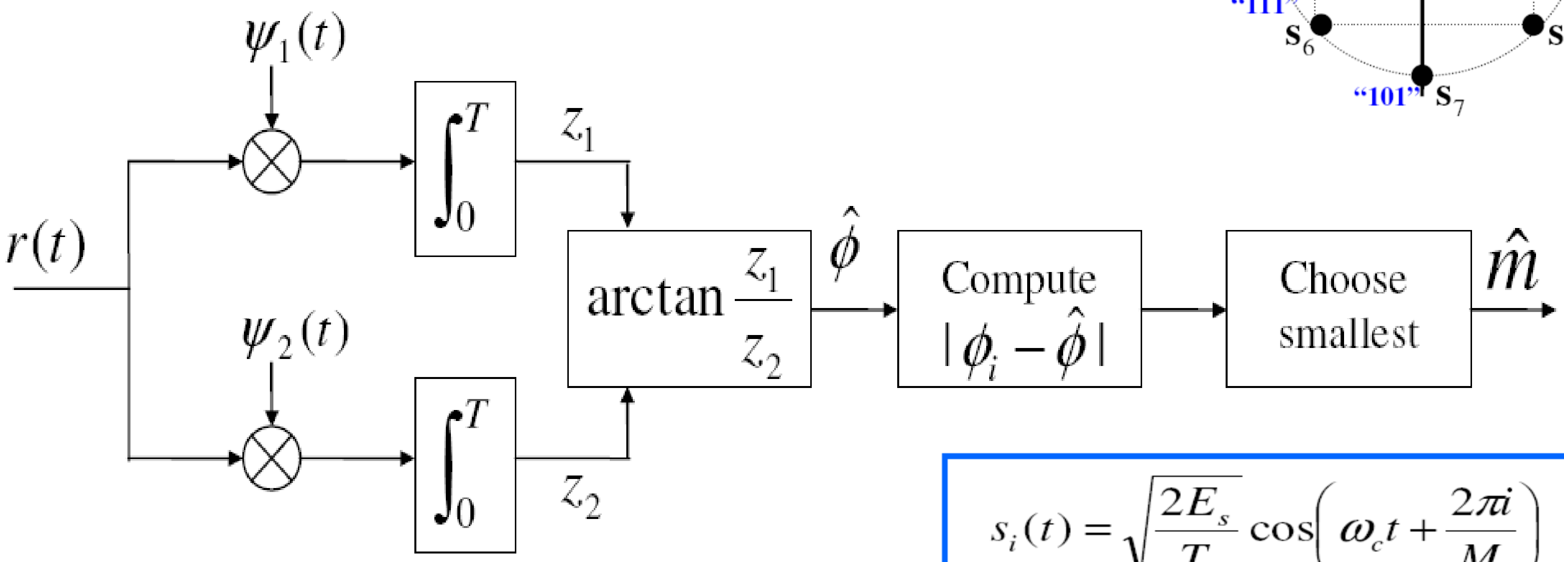
Hence, the use of Equation (5.92) of Chapter 5 yields the average probability of symbol error for coherent M-ary PSK as

$$P_e \simeq \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) \tag{6.47}$$

where it is assumed that $M \geq 4$. The approximation becomes extremely tight, for fixed M, as E/N_0 is increased. For $M = 4$, Equation (6.47) reduces to the same form given in Equation (6.34) for QPSK.

Coherent detection of MPSK





$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos\left(\omega_c t + \frac{2\pi i}{M}\right)$$

$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t) \quad i = 1, \ldots, M$$

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t) \qquad \psi_2(t) = -\sqrt{\frac{2}{T}} \sin(\omega_c t)$$

$$a_{i1} = \sqrt{E_s} \cos\left(\frac{2\pi i}{M}\right) \qquad a_{i2} = \sqrt{E_s} \sin\left(\frac{2\pi i}{M}\right)$$

$$E_s = E_i = \|\mathbf{s}_i\|^2$$

Power Spectra of M-ary PSK Signals

The symbol duration of M-ary PSK is defined by

$$T = T_b \log_2 M \tag{6.48}$$

where T_b is the bit duration. Proceeding in a manner similar to that described for a QPSK signal, we may show that the baseband power spectral density of an M-ary PSK signal is given by

$$\begin{aligned} S_B(f) &= 2E \operatorname{sinc}^2(Tf) \\ &= 2E_b \log_2 M \operatorname{sinc}^2(T_b f \log_2 M) \end{aligned} \tag{6.49}$$

In Figure 6.16, we show the normalized power spectral density $S_B(f)/2E_b$ plotted versus the normalized frequency fT_b for three different values of M, namely, $M = 2, 4, 8$.

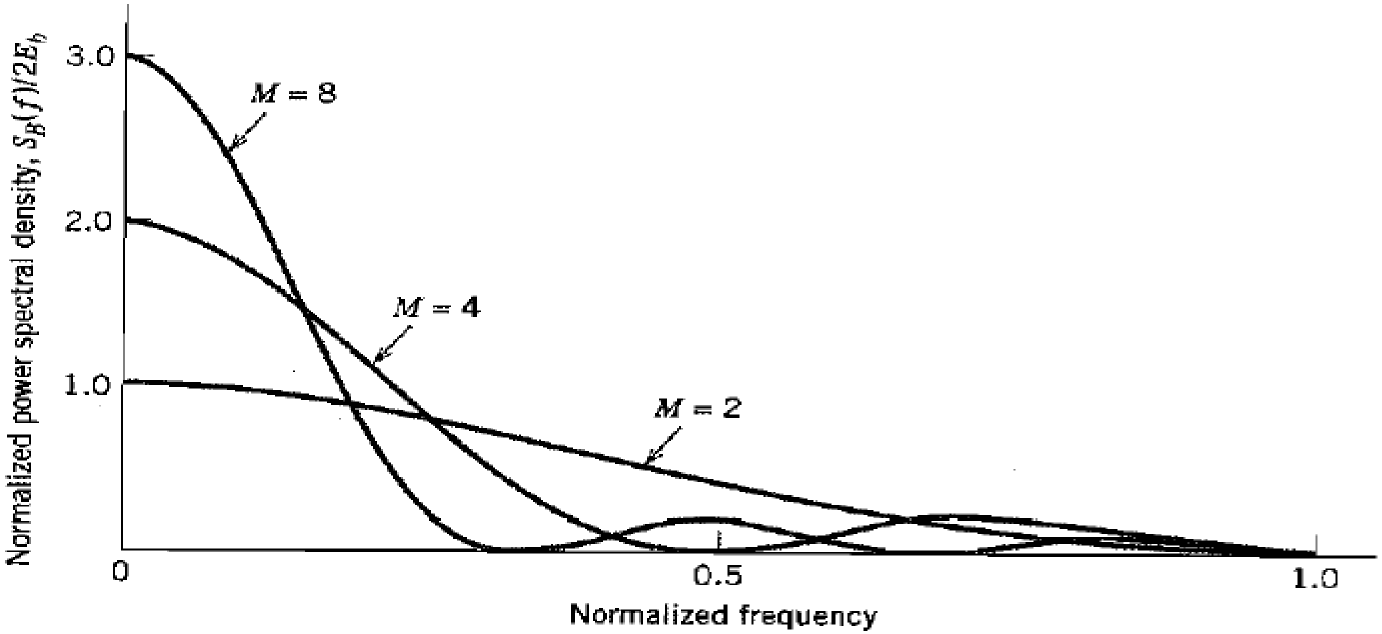


FIGURE 6.16 Power spectra of M-ary PSK signals for $M = 2, 4, 8$.

For the passband basis functions defined in Equations (6.25) and (6.26), the channel bandwidth required to pass M-ary PSK signals (more precisely, the main spectral lobe of M-ary signals) is given by

$$B = \frac{2}{T} \tag{6.50}$$

where T is the symbol duration. But the symbol duration T is related to the bit duration T_b by Equation (6.48). Moreover, the bit rate $R_b = 1/T_b$. Hence, we may redefine the channel bandwidth of Equation (6.50) in terms of the bit rate R_b as

$$B = \frac{2R_b}{\log_2 M} \tag{6.51}$$

Based on this formula, the bandwidth efficiency of M-ary PSK signals is given by

$$\begin{aligned} \rho &= \frac{R_b}{B} \\ &= \frac{\log_2 M}{2} \end{aligned} \tag{6.52}$$

TABLE 6.4 ***Bandwidth efficiency of M-ary PSK signals***

M	2	4	8	16	32	64
ρ (bits/s/Hz)	0.5	1	1.5	2	2.5	3

6.4 Hybrid Amplitude/Phase Modulation Schemes

M-QAM

Two dimensional mod.,... (M-QAM)

- M-ary Quadrature Amplitude Mod. (M-QAM)

$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_c t + \varphi_i)$$

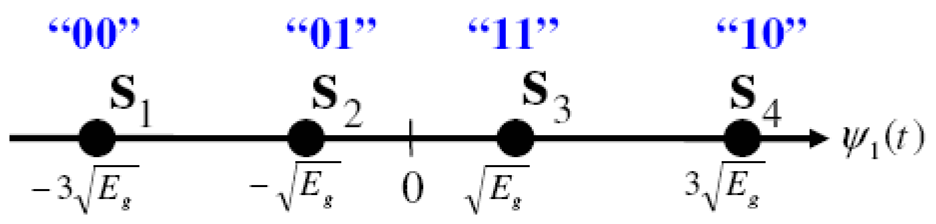
$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t) \quad i = 1, \dots, M$$

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t) \quad \psi_2(t) = \sqrt{\frac{2}{T}} \sin(\omega_c t)$$

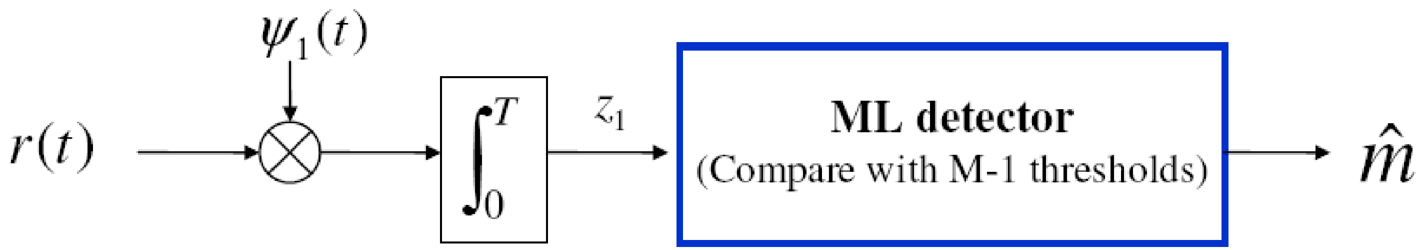
where a_{i1} and a_{i2} are PAM symbols and $E_s = \dfrac{2(M-1)}{3}$

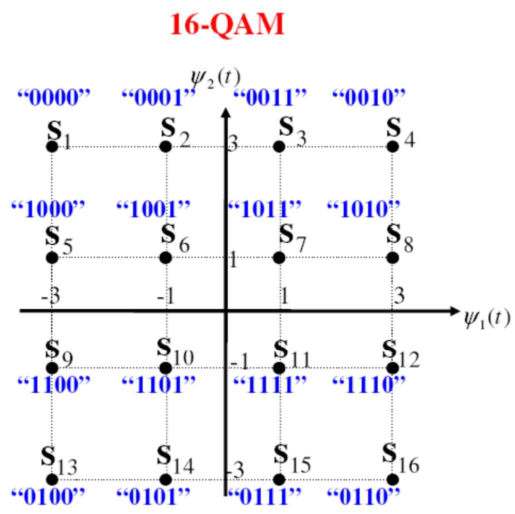
$$(a_{i1}, a_{i2}) = \begin{bmatrix} (-\sqrt{M}+1, \sqrt{M}-1) & (-\sqrt{M}+3, \sqrt{M}-1) & \cdots & (\sqrt{M}-1, \sqrt{M}-1) \\ (-\sqrt{M}+1, \sqrt{M}-3) & (-\sqrt{M}+3, \sqrt{M}-3) & \cdots & (\sqrt{M}-1, \sqrt{M}-3) \\ \vdots & \vdots & \vdots & \vdots \\ (-\sqrt{M}+1, -\sqrt{M}+1) & (-\sqrt{M}+3, -\sqrt{M}+1) & \cdots & (\sqrt{M}-1, -\sqrt{M}+1) \end{bmatrix}$$

4-PAM:

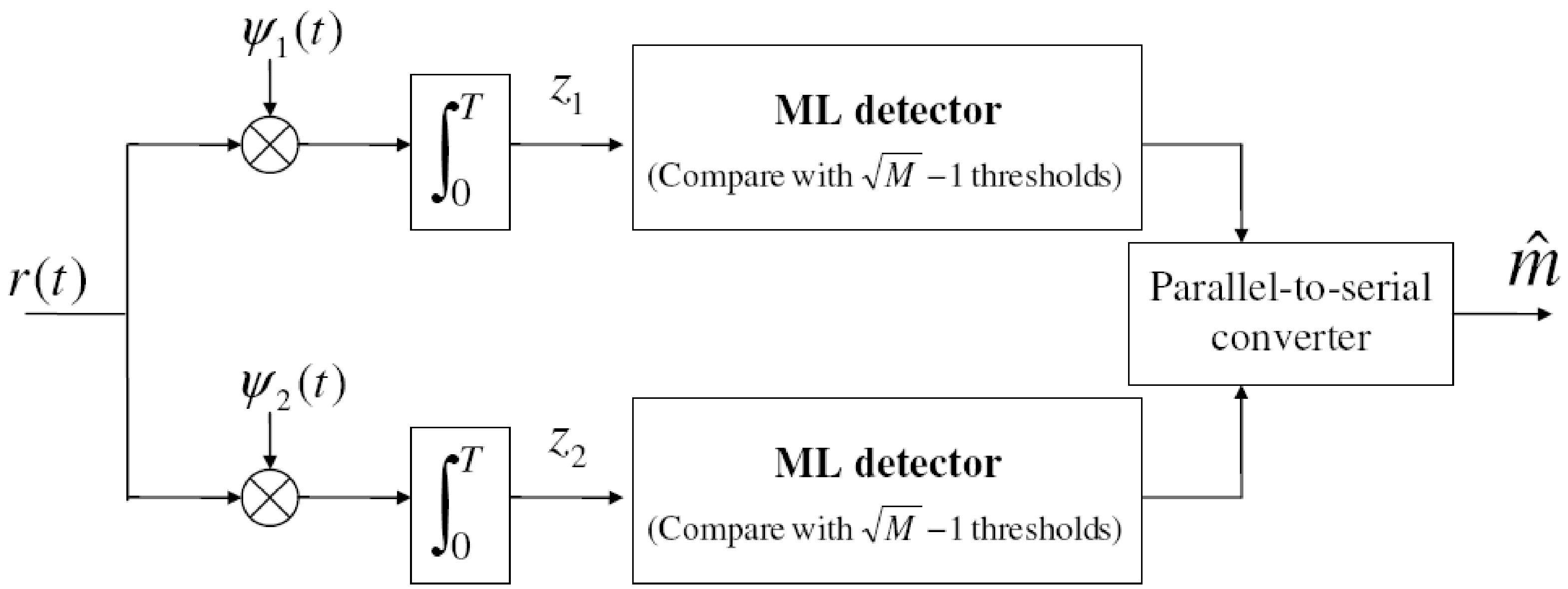


- Coherent detection of M-PAM





- Coherent detection of M-QAM



M-ARY QUADRATURE AMPLITUDE MODULATION

In Chapters 4 and 5, we studied M-ary pulse amplitude modulation (PAM), which is one-dimensional. M-ary QAM is a two-dimensional generalization of M-ary PAM in that its formulation involves two orthogonal passband basis functions, as shown by

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad 0 \le t \le T \tag{6.53}$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t), \qquad 0 \le t \le T \tag{6.54}$$

Let the ith message point s_i in the (ϕ_1, ϕ_2) plane be denoted by $(a_i d_{min}/2, b_i d_{min}/2)$, where d_{min} is the minimum distance between any two message points in the constellation, a_i and b_i are integers, and $i = 1, 2, \ldots, M$. Let $(d_{min}/2) = \sqrt{E_0}$, where E_0 is the energy of the signal with the lowest amplitude. The transmitted M-ary QAM signal for symbol k, say, is then defined by

$$s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_k \sin(2\pi f_c t), \qquad \begin{array}{l} 0 \le t \le T \\ k = 0, \pm 1, \pm 2, \ldots \end{array} \tag{6.55}$$

The signal $s_k(t)$ consists of two phase-quadrature carriers with each one being modulated by a set of discrete amplitudes, hence the name *quadrature amplitude modulation*.

QAM Square Constellations

With an *even* number of bits per symbol, we may write

$$L = \sqrt{M} \tag{6.56}$$

where L is a positive integer. Under this condition, an M-ary QAM square constellation can always be viewed as the *Cartesian product* of a one-dimensional L-ary PAM constellation with itself. By definition, the Cartesian product of two sets of coordinates (representing a pair of one-dimensional constellations) is made up of the set of all possible ordered pairs of coordinates with the first coordinate in each such pair taken from the first set involved in the product and the second coordinate taken from the second set in the product.

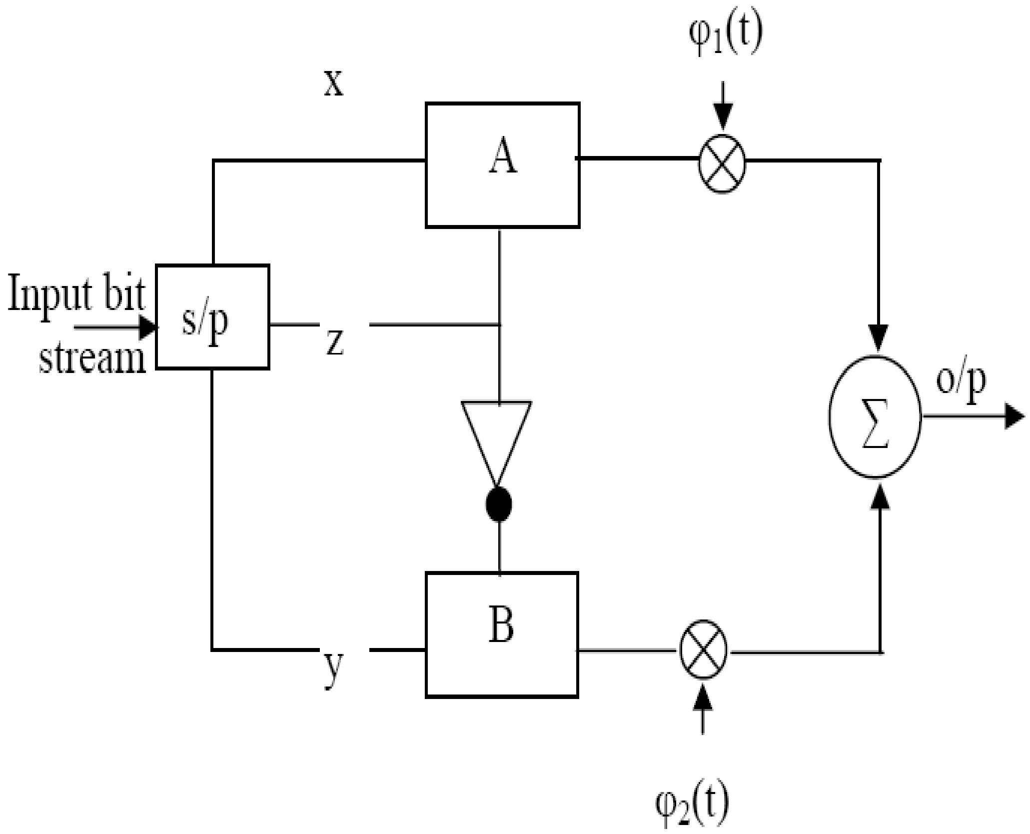
In the case of a QAM square constellation, the ordered pairs of coordinates naturally form a square matrix, as shown by

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\ \vdots & \vdots & & \vdots \\ (-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1) \end{bmatrix} \tag{6.57}$$

Examples

Q1. In the shown transmitter the amplifiers A& B are controlled by a control bit "*z*". If "*z*" is '1' the amplification ratio (for A and B) is 2:1 and if "*z*" is '-1' the ratio (for A and B)is 1:1. The input stream is divided into symbols each of 3 bits designated as xyz in order. The bits are represented using polar NRZ format with =5v and –A=-5v.

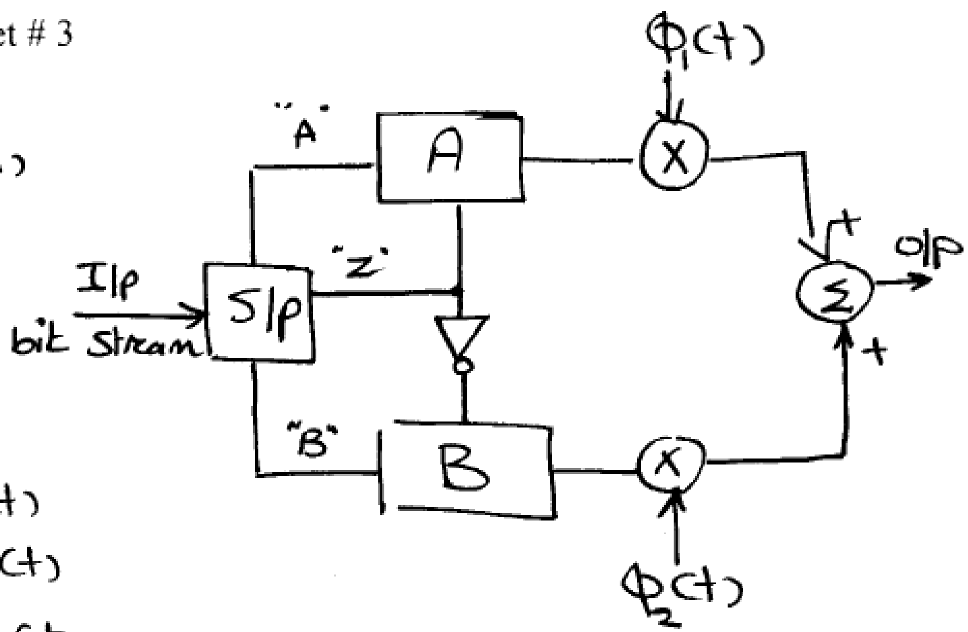
1. Find all possible outputs of the transmitter in terms of $\phi 1$ and $\phi 2$.
2. Sketch to scale the signals in Signal space. and define the Decision Regions (DR) and the Decision Boundaries(DB).

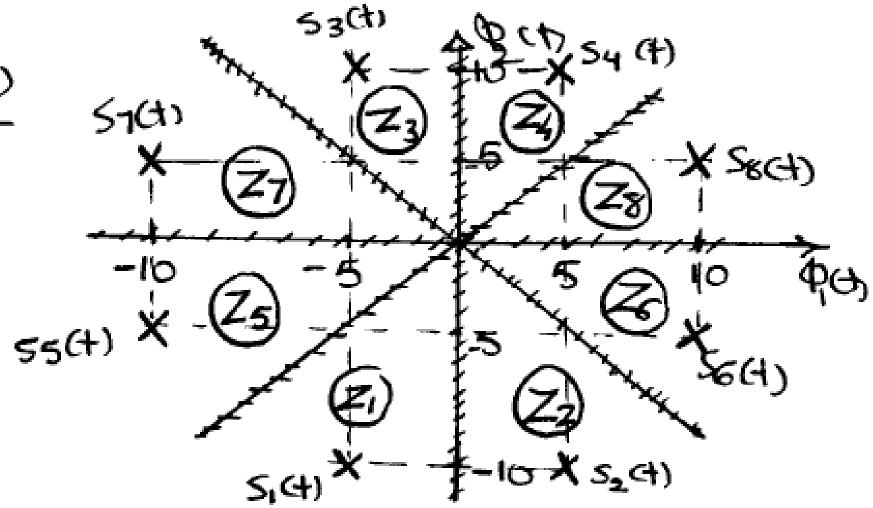


Q1

1) Possible o/ps

"Z"	"B"	"A"	o/p
0	0	0	$s_1(t) = -5\phi_1(t) - 10\phi_2(t)$
0	0	1	$s_2(t) = 5\phi_1(t) - 10\phi_2(t)$
0	1	0	$s_3(t) = -5\phi_1(t) + 10\phi_2(t)$
0	1	1	$s_4(t) = 5\phi_1(t) + 10\phi_2(t)$
1	0	0	$s_5(t) = -10\phi_1(t) - 5\phi_2(t)$
1	0	1	$s_6(t) = 10\phi_1(t) - 5\phi_2(t)$
1	1	0	$s_7(t) = -10\phi_1(t) + 5\phi_2(t)$
1	1	1	$s_8(t) = 10\phi_1(t) + 5\phi_2(t)$





2)

Q2. A communication system uses a signal $s_1(t) = 3\cos(200\pi t) \quad 0 \le t \le 2\sec$ to represent the digit '1'.

To present the digit '0' either $s_2(t)$ or $s_2'(t)$ is available, where

$s_2(t) = -4\cos(200\pi t) \qquad s_2'(t) = 4\cos(400\pi t) \quad 0 \le t \le 2\sec$.

The noise is assumed to be AWGN with two-sided PSD= $\frac{N_0}{2}$ =2watt/Hz.

1. Sketch to scale the two cases in S.S. showing the DRs and the DBs.
2. Calculate the minimum average probability of error.
3. Show that the receiver in both cases can be implemented using a single arm receiver and define each part of the receiver.

Q2.

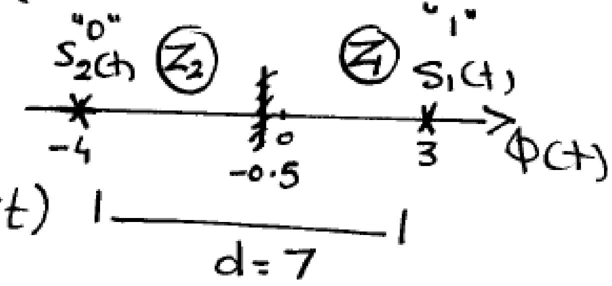
Case (I)

1) $s_1(t) = 3\cos(200\pi t)$ $\quad 0 \le t \le 2\text{ sec}$

$s_2(t) = -4\cos(200\pi t)$

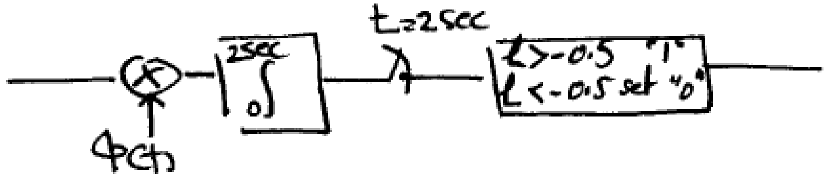
$f_{c1} = f_{c2} = f_c = 100\text{ Hz}$

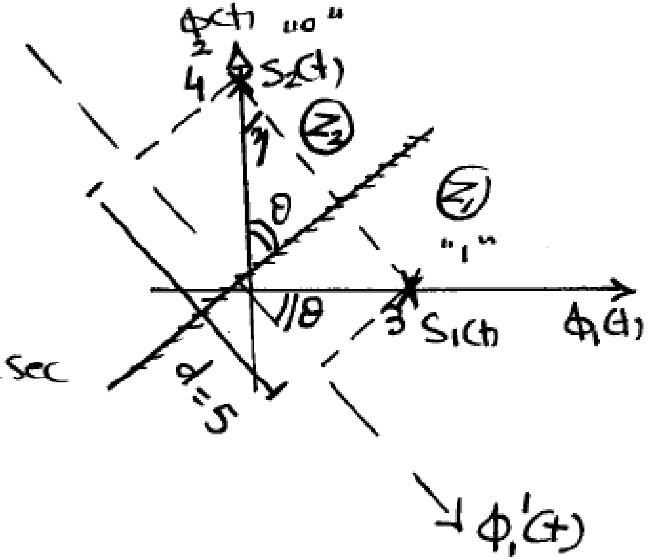
$\phi(t) = \sqrt{\frac{2}{T}}\cos(200\pi t) = \cos(200\pi t)$



2) $P_a[\varepsilon] = P_e = Q\left[\frac{d}{\sqrt{2N_0}}\right] = Q\left[\frac{7}{\sqrt{8}}\right]$

3)





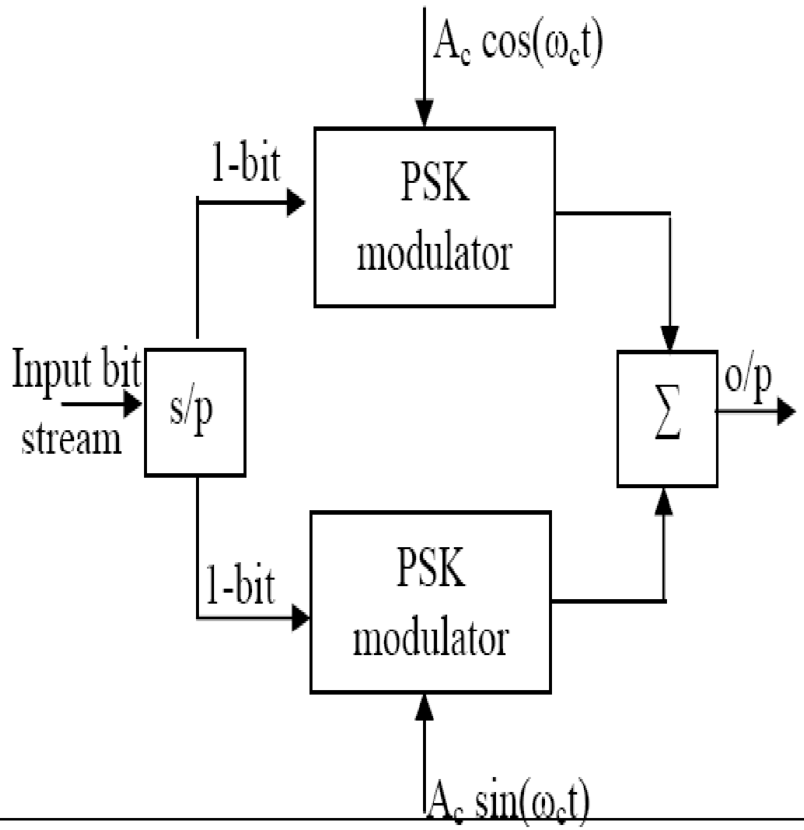
Case (II)

1) $s_1(t) = 3\cos(200\pi t)$ $\quad 0 \le t \le 2\text{ sec}$

$s_2'(t) = 4\cos(400\pi t)$

Q3. The below digital modulator scheme produces 4 equally likely messages.

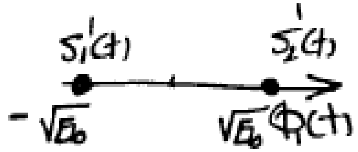
1. Sketch the output possible signals in SS.
2. Draw the DRs and DBs.
3. Calculate the average energy.
4. Calculate the minimum average probability of error if the noise is assumed to be AWGN of zero mean and PSD= $\frac{N_0}{2}$

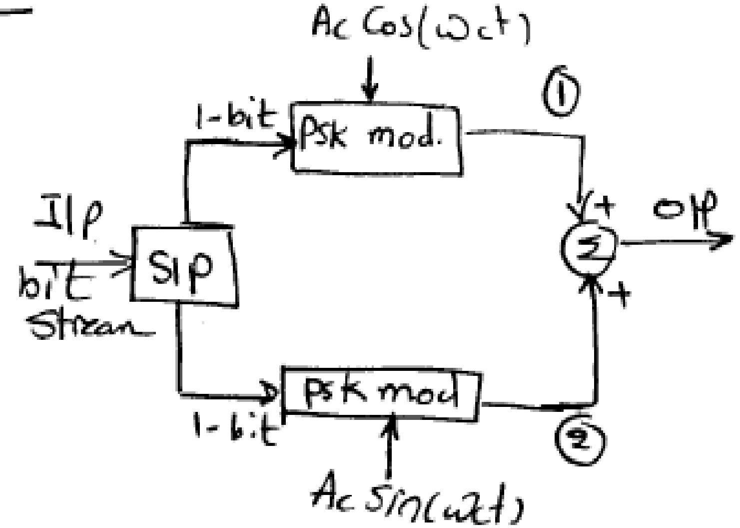


Q4 o/p at pt. ①

$s_1'(t) = -\sqrt{E_b}\,\phi_1(t)$

$s_2'(t) = \sqrt{E_b}\,\phi_1(t)$

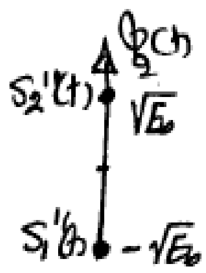




o/p at Pt. ②

$s_1''(t) = -\sqrt{E_b}\,\phi_2(t)$

$s_2''(t) = \sqrt{E_b}\,\phi_2(t)$



Tx. O/P

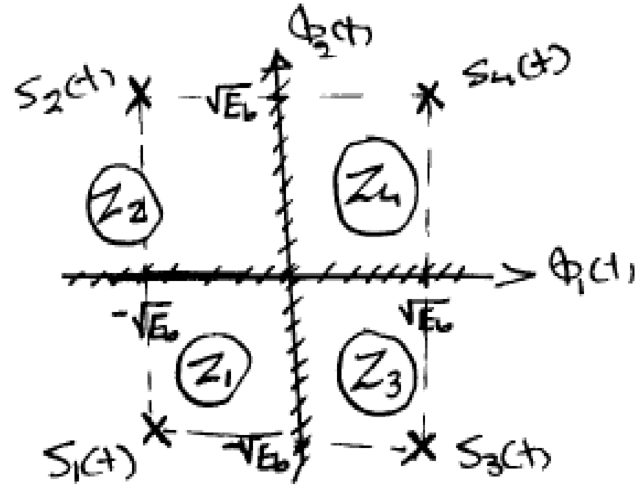
$s_1(t) = -\sqrt{E_b}\,\phi_1(t) - \sqrt{E_b}\,\phi_2(t)$

$s_2(t) = -\sqrt{E_b}\,\phi_1(t) + \sqrt{E_b}\,\phi_2(t)$

$s_3(t) = \sqrt{E_b}\,\phi_1(t) - \sqrt{E_b}\,\phi_2(t)$

$s_4(t) = \sqrt{E_b}\,\phi_1(t) + \sqrt{E_b}\,\phi_2(t)$

2)



3) $E_{av} = \dfrac{4(E_b + E_b)}{4} = 2E_b$ Joule

4) 2-Sided D.R(s) (Z1), (Z2), (Z3), (Z4)

$$P_e = Q\left[\frac{d}{\sqrt{2N_0}}\right] = Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

$$P[c|m_i] = (1-P_e)^2$$

$$P_{av}[c] = \frac{1}{K}\left[K\left(1-2P_e+P_e^2\right)\right]$$

$$P_{av}[\varepsilon] = 1-P_{av}[c] = 2P_e - P_e^2$$

$$P_{av}[\varepsilon] = 2Q\left[\sqrt{\frac{2E_b}{N_0}}\right] - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2.$$

▶ EXAMPLE 6.3

Consider a 16-QAM whose signal constellation is depicted in Figure 6.17*a*. The encoding of the message points shown in this figure is as follows:

- Two of the four bits, namely, the left-most two bits, specify the quadrant in the (ϕ_1, ϕ_2)-plane in which a message point lies. Thus, starting from the first quadrant and proceeding counterclockwise, the four quadrants are represented by the dibits 11, 10, 00, and 01.
- The remaining two bits are used to represent one of the four possible symbols lying within each quadrant of the (ϕ_1, ϕ_2)-plane.

Note that the encoding of the four quadrants and also the encoding of the symbols in each quadrant follow the Gray coding rule.

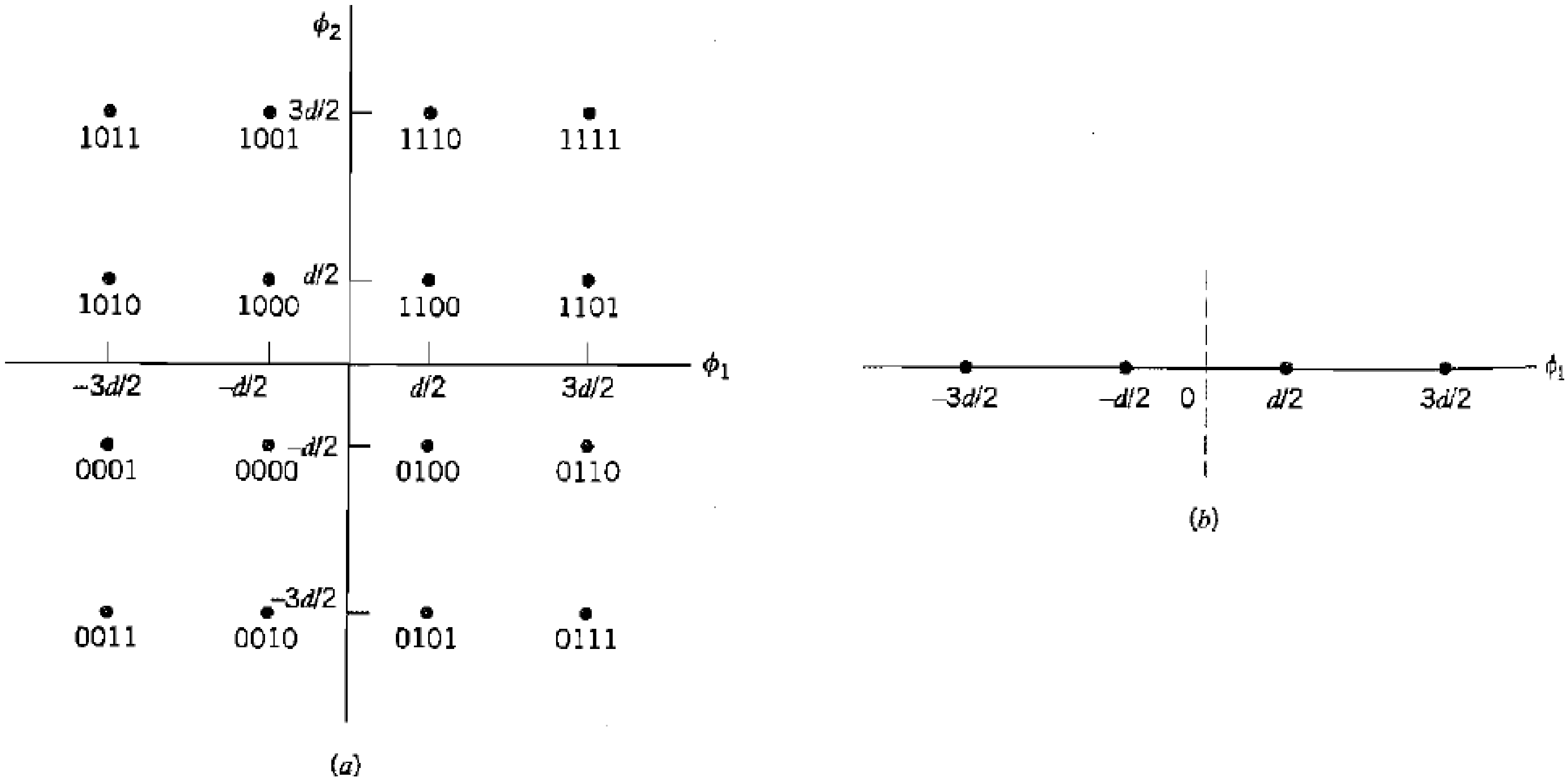


FIGURE 6.17 (*a*) Signal-space diagram of *M*-ary QAM for $M = 16$; the message points in each quadrant are identified with Gray-encoded quadbits. (*b*) Signal-space diagram of the corresponding 4-PAM signal.

For the example at hand, we have $L = 4$. Thus the square constellation of Figure 6.17*a* is the Cartesian product of the 4-PAM constellation shown in Figure 6.17*b* with itself. Moreover, the matrix of Equation (6.57) has the value

$$\{a_i, b_i\} = \begin{bmatrix} (-3, 3) & (-1, 3) & (1, 3) & (3, 3) \\ (-3, 1) & (-1, 1) & (1, 1) & (3, 1) \\ (-3, -1) & (-1, -1) & (1, -1) & (3, -1) \\ (-3, -3) & (-1, -3) & (1, -3) & (3, -3) \end{bmatrix}$$

◀

To calculate the probability of symbol error for M-ary QAM, we exploit the property that a QAM square constellation can be factored into the product of the corresponding PAM constellation with itself. We may thus proceed as follows:

1. The probability of correct detection for M-ary QAM may be written as

$$P_c = (1 - P_e')^2 \tag{6.58}$$

where P_e' is the probability of symbol error for the corresponding L-ary PAM with $L = \sqrt{M}$.

2. The probability of symbol error P_e' is defined by

$$P_e' = \left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \tag{6.59}$$

(Note that $L = \sqrt{M}$ in the M-ary QAM corresponds to M in the M-ary PAM considered in Problem 4.27.)

3. The probability of symbol error for M-ary QAM is given by

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - (1 - P'_e)^2 \\ &\simeq 2P'_e \end{aligned} \tag{6.60}$$

where it is assumed that P'_e is small enough compared to unity to justify ignoring the quadratic term.

Hence, using Equations (6.58) and (6.59) in Equation (6.60), we find that the probability of symbol error for M-ary QAM is approximately given by

$$P_e \simeq 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \tag{6.61}$$

The transmitted energy in M-ary QAM is variable in that its instantaneous value depends on the particular symbol transmitted. It is therefore more logical to express P_e in terms of the *average* value of the transmitted energy rather than E_0. Assuming that the L amplitude levels of the in-phase or quadrature component are equally likely, we have

$$E_{av} = 2\left[\frac{2E_0}{L} \sum_{i=1}^{L/2} (2i - 1)^2\right] \tag{6.62}$$

where the multiplying factor of 2 outside the square brackets accounts for the equal contributions made by the in-phase and quadrature components. The limits of the summation and the multiplying factor of 2 inside the square brackets take account of the symmetric nature of the pertinent amplitude levels around zero. Summing the series in Equation (6.62), we get

$$\begin{aligned} E_{\text{av}} &= \frac{2(L^2 - 1)E_0}{3} \\ &= \frac{2(M - 1)E_0}{3} \end{aligned} \tag{6.63}$$

Accordingly, we may rewrite Equation (6.61) in terms of E_{av} as

$$P_e \simeq 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{2(M - 1)N_0}}\right) \tag{6.64}$$

which is the desired result.

The case of $M = 4$ is of special interest. The signal constellation for this value of M is the same as that for QPSK. Indeed, putting $M = 4$ in Equation (6.64) and noting that for this special case E_{av} equals E, where E is the energy per symbol, we find that the resulting formula for the probability of symbol error becomes identical to that in Equation (6.34), and so it should.