Week 5-P2

5. Signal-Space Analysis

5.2 Geometric Representation of Signals

Chapter 5 Signal - Space Analysis

5.1 Introdution

Figure 5.1 ***Block diagram of a generic digital communication system.***

The message source emits one symbol m_i

$$m_i \in \{m_1, m_2, \ldots, m_M\}$$

The probability the symbol m_i is emitted is

$$P_i = P(m_i) = \frac{1}{M} \text{ for } i = 1,2,\ldots,M \tag{5.1}$$

The transmitter codes m_i into $s_i(t)$

The energy of $s_i(t)$ is

$$E_i = \int_0^T s_i^2(t)dt \;, i = 1,2,\ldots,M \tag{5.2}$$

Assuming that the channel is linear and the channel noise , $w(t)$,is AWGN

$$x(t) = s_i(t) + w(t) \quad , \text{for } 0 \le t \le T \text{ and } i = 1,2,..,M \quad (5.3)$$

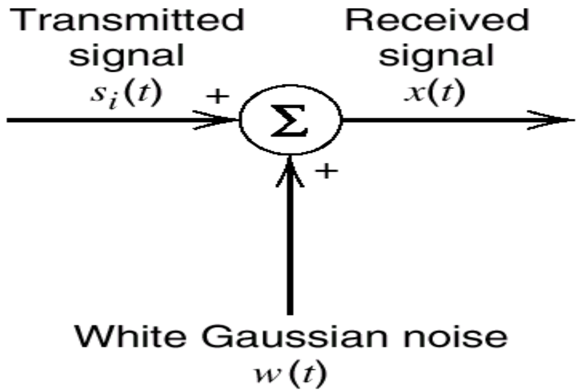

Figure 5.2 ***Additive white Gaussian noise (AWGN) model of a channel.***

At the receiver the average prob. of symbol error

$$\text{Pe} = \sum_{i=1}^{M} p_i P(\hat{m} \ne m_i | m_i) \qquad (5.4)$$

m_i : the transmitted symbol

$\hat{m}$: the estimate

Geometric Representation of Modulation Signals

The essence of *geometric representation of signals*[1] is to represent any set of M energy signals $\{s_i(t)\}$ as linear combinations of N *orthonormal basis functions*, where $N \le M$. That is to say, given a set of real-valued energy signals $s_1(t), s_2(t), \ldots, s_M(t)$, each of duration T seconds, we write

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \tag{5.5}$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\, dt, \qquad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases} \tag{5.6}$$

The real-valued basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are *orthonormal*, by which we mean

$$\int_0^T \phi_i(t)\phi_j(t)\, dt = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j \end{cases} \tag{5.7}$$

where δ_{ij} is the *Kronecker delta*. The first condition of Equation (5.7) states that each basis function is *normalized* to have unit energy. The second condition states that the basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are *orthogonal* with respect to each other over the interval $0 \le t \le T$.

Accordingly, we may state that each signal in the set $\{s_i(t)\}$ is completely determined by the *vector* of its coefficients

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \qquad i = 1, 2, \ldots, M \tag{5.8}$$

The vector $\mathbf{s}_i$ is called a *signal vector*. Furthermore, if we conceptually extend our conventional notion of two- and three-dimensional Euclidean spaces to an *N-dimensional Euclidean space*, we may visualize the set of signal vectors $\{\mathbf{s}_i | i = 1, 2, \ldots, M\}$ as defining a corresponding set of M points in an N-dimensional Euclidean space, with N mutually perpendicular axes labeled $\phi_1, \phi_2, \ldots, \phi_N$. This N-dimensional Euclidean space is called the *signal space*.

$$\| \mathbf{s}_i \|^2 = \mathbf{s}_i^T \mathbf{s}_i = \sum_{j=1}^{N} s_{ij}^2, \qquad i = 1, 2, \ldots, M$$

$$E_i = \int_0^T s_i^2(t)\, dt$$

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t) \right] \left[\sum_{k=1}^{N} s_{ik}\phi_k(t) \right] dt$$

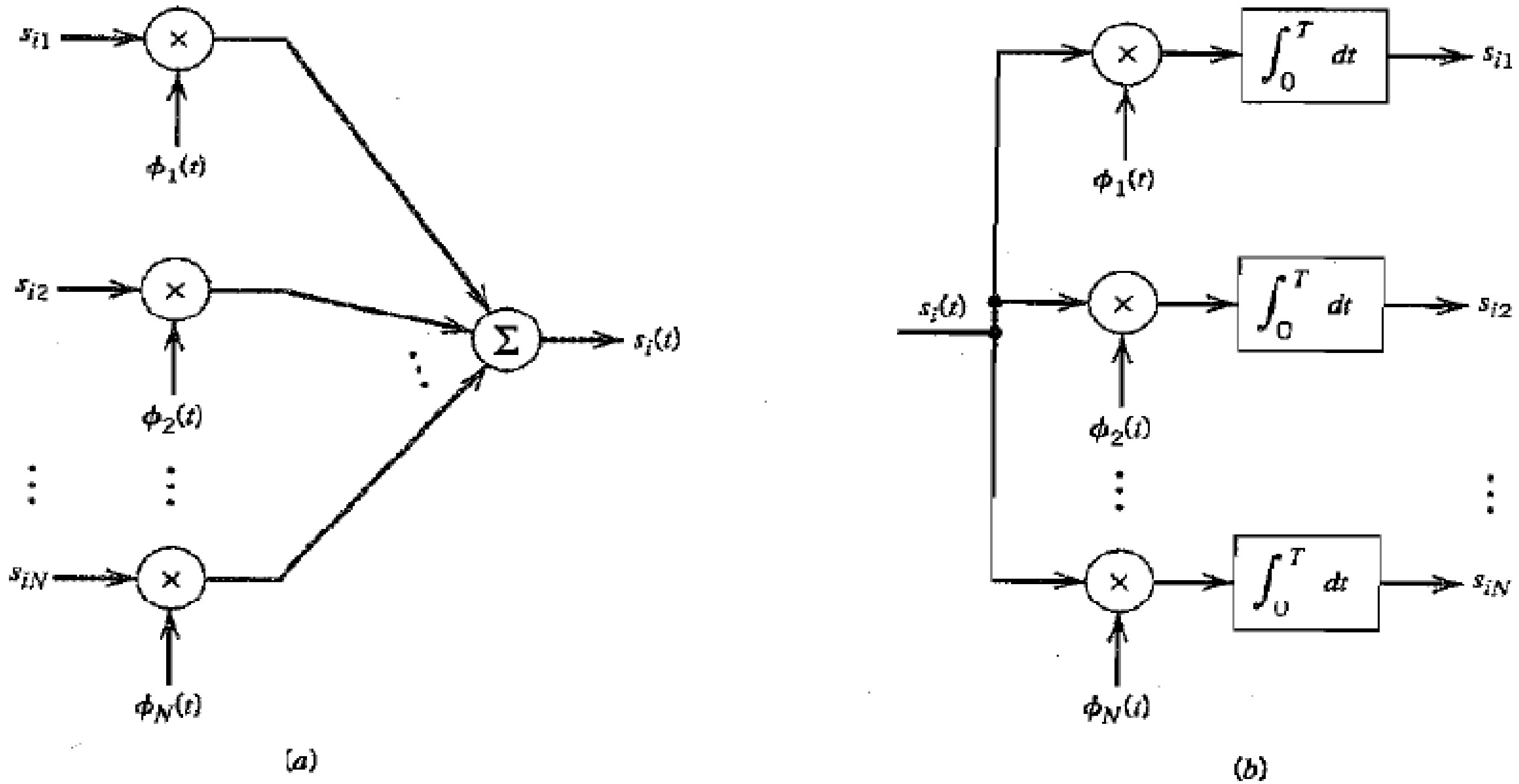

FIGURE 5.3 (*a*) Synthesizer for generating the signal $s_i(t)$. (*b*) Analyzer for generating the set of signal vectors $\{s_i\}$.

the energy of a signal $s_i(t)$ is equal to the squared length of the signal vector $s_i(t)$

$$E_i = \sum_{j=1}^{N} s_{ij}^2 = \| \mathbf{s}_i \|^2$$

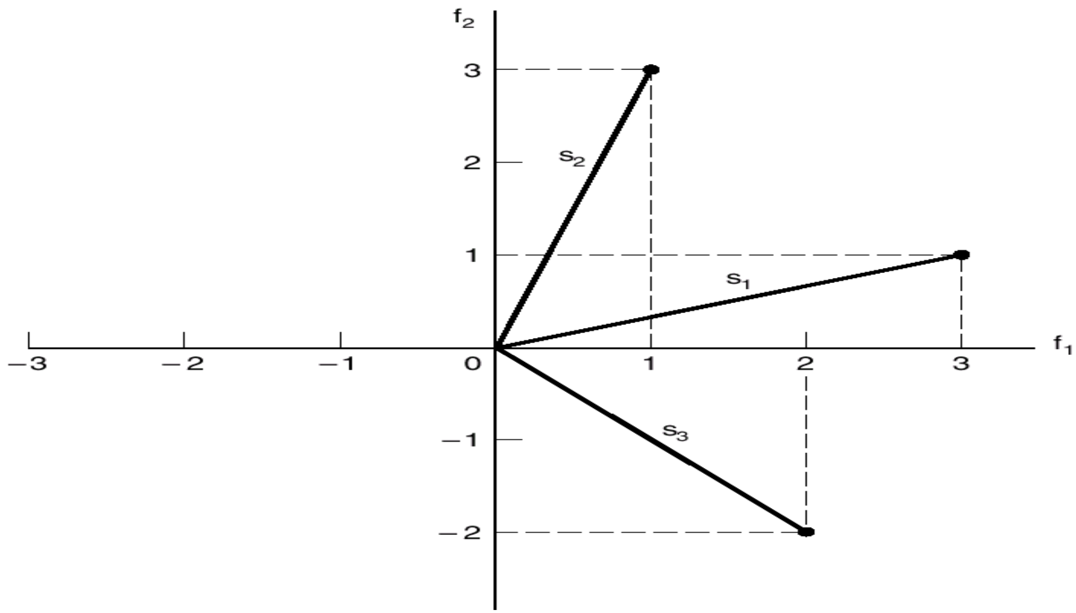

Figure 5.4
Illustrating the geometric representation of signals for the case when $N = 2$ and $M = 3$.

The idea of visualizing a set of energy signals geometrically, as just described, is of profound importance. It provides the mathematical basis for the geometric representation of energy signals, thereby paving the way for the noise analysis of digital communication systems in a conceptually satisfying manner. This form of representation is illustrated in Figure 5.4 for the case of a two-dimensional signal space with three signals, that is, $N = 2$ and $M = 3$.

In an N-dimensional Euclidean space, we may define *lengths* of vectors and *angles* between vectors. It is customary to denote the length (also called the *absolute value* or *norm*) of a signal vector $\mathbf{s}_i$ by the symbol $\|\mathbf{s}_i\|$. The squared-length of any signal vector $\mathbf{s}_i$ is defined to be the *inner product* or *dot product* of $\mathbf{s}_i$ with itself, as shown by

$$\begin{aligned} \|\mathbf{s}_i\|^2 &= \mathbf{s}_i^T \mathbf{s}_i \\ &= \sum_{j=1}^{N} s_{ij}^2, \qquad i = 1, 2, \ldots, M \end{aligned} \tag{5.9}$$

where s_{ij} is the jth element of $\mathbf{s}_i$, and the superscript T denotes matrix transposition.

There is an interesting relationship between the energy content of a signal and its representation as a vector. By definition, the energy of a signal $s_i(t)$ of duration T seconds is

$$E_i = \int_0^T s_i^2(t)\, dt \tag{5.10}$$

Therefore, substituting Equation (5.5) into (5.10), we get

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\phi_k(t)\right] dt$$

Interchanging the order of summation and integration, and then rearranging terms, we get

$$E_i = \sum_{j=1}^{N} \sum_{k=1}^{N} s_{ij} s_{ik} \int_0^T \phi_j(t)\phi_k(t)dt \tag{5.11}$$

But since the $\phi_j(t)$ form an orthonormal set, in accordance with the two conditions of Equation (5.7), we find that Equation (5.11) reduces simply to

$$\begin{aligned} E_i &= \sum_{j=1}^{N} s_{ij}^2 \\ &= \| \mathbf{s}_i \|^2 \end{aligned} \tag{5.12}$$

Thus Equations (5.9) and (5.12) show that the energy of a signal $s_i(t)$ is equal to the squared length of the signal vector $\mathbf{s}_i(t)$ representing it.

In the case of a pair of signals $s_i(t)$ and $s_k(t)$, represented by the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$, respectively, we may also show that

$$\int_0^T s_i(t)s_k(t)\,dt = \mathbf{s}_i^T \mathbf{s}_k \tag{5.13}$$

Equation (5.13) states that the *inner product* of the signals $s_i(t)$ and $s_k(t)$ over the interval [0, T], using their time-domain representations, is equal to the inner product of their respective vector representations $\mathbf{s}_i$ and $\mathbf{s}_k$. Note that the inner product of $s_i(t)$ and $s_k(t)$ is *invariant* to the choice of basis functions $\{\phi_j(t)\}_{j=1}^{N}$ in that it only depends on the components of the signals $s_i(t)$ and $s_k(t)$ projected onto each of the basis functions.

Yet another useful relation involving the vector representations of the signals $s_i(t)$ and $s_k(t)$ is described by

$$\begin{aligned} \| \mathbf{s}_i - \mathbf{s}_k \|^2 &= \sum_{j=1}^{N} (s_{ij} - s_{kj})^2 \\ &= \int_0^T (s_i(t) - s_k(t))^2 dt \end{aligned} \tag{5.14}$$

where $\| \mathbf{s}_i - \mathbf{s}_k \|$ is the *Euclidean distance*, d_{ik}, between the points represented by the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$.

To complete the geometric representation of energy signals, we need to have a representation for the angle θ_{ik} subtended between two signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$. By definition, the *cosine of the angle* θ_{ik} is equal to the inner product of these two vectors divided by the product of their individual norms, as shown by

$$\cos \theta_{ik} = \frac{\mathbf{s}_i^T \mathbf{s}_k}{\| \mathbf{s}_i \| \, \| \mathbf{s}_k \|} \tag{5.15}$$

The two vectors $\mathbf{s}_i$ and $\mathbf{s}_k$ are thus *orthogonal* or *perpendicular* to each other if their inner product $\mathbf{s}_i^T \mathbf{s}_k$ is zero, in which case $\theta_{ik} = 90$ degrees; this condition is intuitively satisfying.

$$\| \mathbf{s}_i - \mathbf{s}_k \|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2 = \int_0^T (s_i(t) - s_k(t))^2 dt \qquad (5.14)$$

where $\| \mathbf{s}_i - \mathbf{s}_k \|$ is the *Euclidean distance*, d_{ik}, between the points represented by the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$.

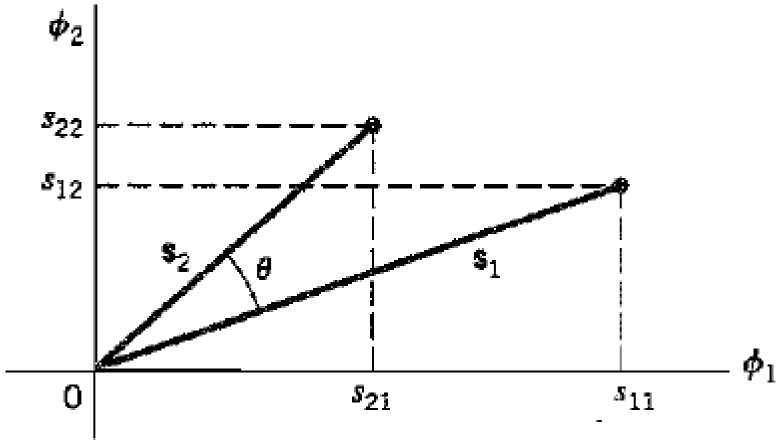

FIGURE 5.5 Vector representations of signals $s_1(t)$ and $s_2(t)$, providing the background picture for proving the Schwarz inequality.

Gram-Schmidt Orthogonalization Procedure

Having demonstrated the elegance of the geometric representation of energy signals, how do we justify it in mathematical terms? The answer lies in the *Gram-Schmidt orthogonalization procedure*, for which we need a *complete orthonormal set of basis functions*. To proceed with the formulation of this procedure, suppose we have a set of M energy signals denoted by $s_1(t), s_2(t), \ldots, s_M(t)$. Starting with $s_1(t)$ chosen from this set arbitrarily, the first basis function is defined by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \tag{5.19}$$

where E_1 is the energy of the signal $s_1(t)$. Then, clearly, we have

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned} \tag{5.20}$$

where the coefficient $s_{11} = \sqrt{E_1}$ and $\phi_1(t)$ has unit energy, as required.

Next, using the signal $s_2(t)$, we define the coefficient s_{21} as

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt \tag{5.21}$$

We may thus introduce a new intermediate function

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \tag{5.22}$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$ by virtue of Equation (5.21) and the fact that the basis function $\phi_1(t)$ has unit energy. Now, we are ready to define the second basis function as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} \tag{5.23}$$

Substituting Equation (5.22) into (5.23) and simplifying, we get the desired result

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \tag{5.24}$$

where E_2 is the energy of the signal $s_2(t)$. It is clear from Equation (5.23) that

$$\int_0^T \phi_2^2(t)\,dt = 1$$

and from Equation (5.24) that

$$\int_0^T \phi_1(t)\phi_2(t)\,dt = 0$$

That is to say, $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal pair, as required.

Continuing in this fashion, we may in general define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t) \tag{5.25}$$

where the coefficients s_{ij} are themselves defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \qquad j = 1, 2, \ldots, i-1 \tag{5.26}$$

Equation (5.22) is a special case of Equation (5.25) with $i = 2$. Note also that for $i = 1$, the function $g_i(t)$ reduces to $s_i(t)$.

Given the $g_i(t)$, we may now define the set of basis functions

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}}, \qquad i = 1, 2, \ldots, N \tag{5.27}$$

which form an orthonormal set. The dimension N is less than or equal to the number of given signals, M, depending on one of two possibilities:

- The signals $s_1(t), s_2(t), \ldots, s_M(t)$ form a *linearly independent set*, in which case $N = M$.
- The signals $s_1(t), s_2(t), \ldots, s_M(t)$ are *not* linearly independent, in which case $N < M$, and the intermediate function $g_i(t)$ is zero for $i > N$.

Note that the conventional Fourier series expansion of a periodic signal is an example of a particular expansion of the type described herein. Also, the representation of a band-limited signal in terms of its samples taken at the Nyquist rate may be viewed as another sample of a particular expansion of this type. However, two important distinctions should be made:

1. The form of the basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ has not been specified. That is to say, unlike the Fourier series expansion of a periodic signal or the sampled representation of a band-limited signal, we have not restricted the Gram-Schmidt orthogonalization procedure to be in terms of sinusoidal functions or sinc functions of time.
2. The expansion of the signal $s_i(t)$ in terms of a finite number of terms is not an approximation wherein only the first N terms are significant but rather an *exact* expression where N and only N terms are significant.