372 assignment #1

TABLE 3.1
Short Table of Fourier Transforms

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\frac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\frac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\frac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\frac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\frac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\frac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\frac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\frac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}\,(\pi f\tau)$	
18	$2B\,\mathrm{sinc}\,(2\pi Bt)$	$\Pi\left(\frac{f}{2B}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\pi f\tau}{2}\right)$	
20	$B\,\mathrm{sinc}^2\,(\pi Bt)$	$\Delta\left(\frac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\frac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-2(\sigma\pi f)^2}$	

TABLE 3.2
Properties of Fourier Transform Operations

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t)+g_2(t)$	$G_1(f)+G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\|a\|}G\left(\frac{f}{a}\right)$
Time shifting	$g(t-t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f-f_0)$
Time convolution	$g_1(t)*g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f)*G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(f)}{j2\pi f}+\frac{1}{2}G(0)\delta(f)$

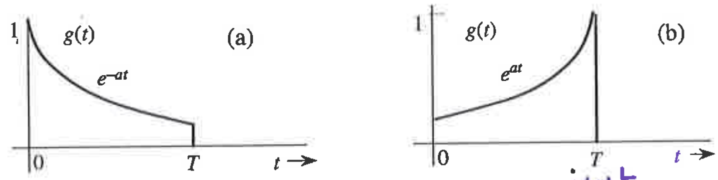
This is the trigonometric form of the (inverse) Fourier transform.

(b) Express the Fourier integral (inverse Fourier transform) for $g(t) = e^{-at}u(t)$ in the trigonometric form given in part **(a)**.

3.1-3 If $g(t) \Longleftrightarrow G(f)$, then show that $g^*(t) \Longleftrightarrow G^*(-f)$.

3.1-4 From definition (3.9a), find the Fourier transforms of the signals shown in Fig. P3.1-4.

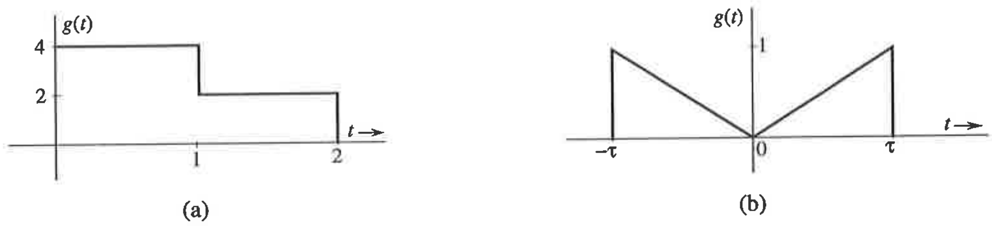
Figure P.3.1-4



$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$

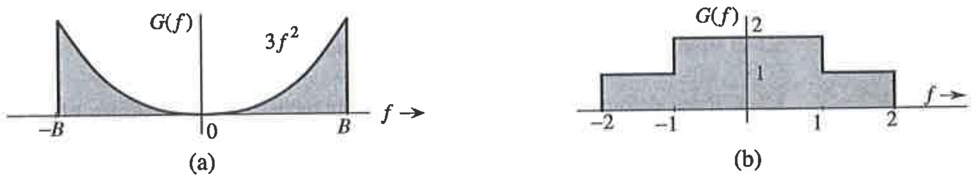
3.1-5 From definition (3.9a), find the Fourier transforms of the signals shown in Fig. P3.1-5.

Figure P.3.1-5



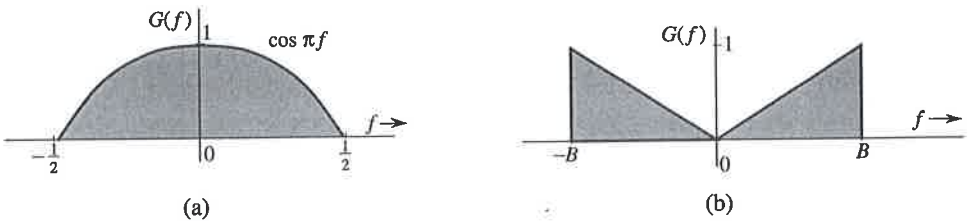
3.1-6 From definition (3.9b), find the inverse Fourier transforms of the spectra shown in Fig. P3.1-6.

Figure P.3.1-6



3.1-7 From definition (3.9b), find the inverse Fourier transforms of the spectra shown in Fig. P3.1-7.

Figure P.3.1-7



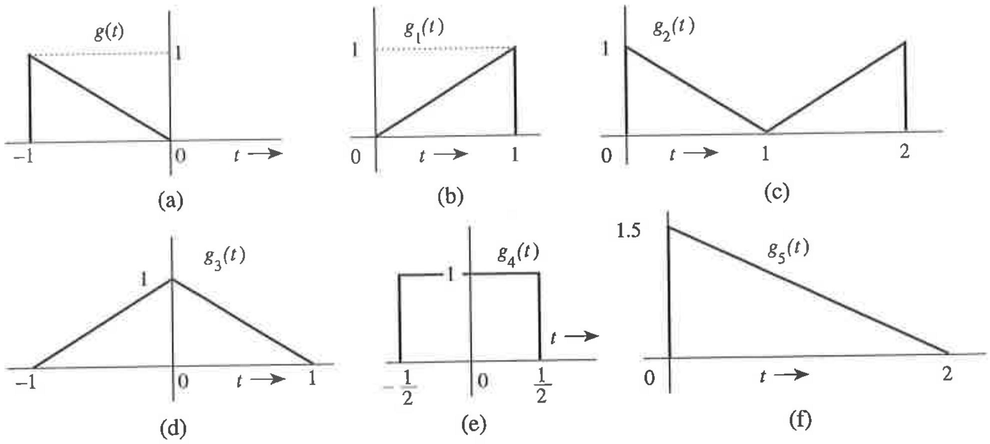
3.3-2 The Fourier transform of the triangular pulse $g(t)$ in Fig. P3.3-2a is given as

$$G(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$$

Use this information, and the time-shifting and time-scaling properties, to find the Fourier transforms of the signals shown in Fig. P3.3-2b–f.

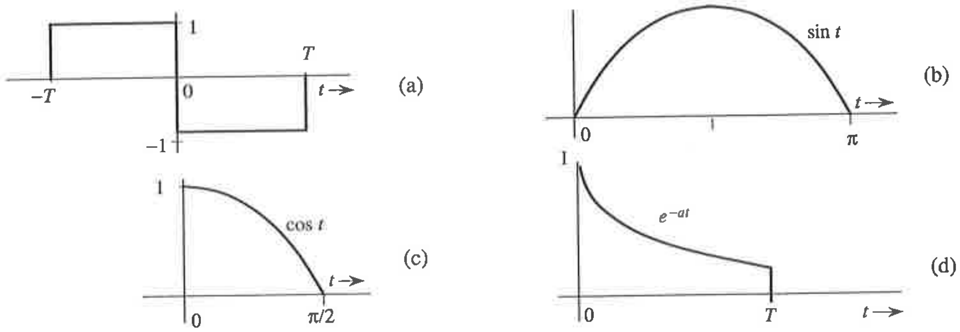
Hint: Time inversion in $g(t)$ results in the pulse $g_1(t)$ in Fig. P3.3-2b; consequently $g_1(t) = g(-t)$. The pulse in Fig. P3.3-2c can be expressed as $g(t-T)+g_1(t-T)$ [the sum of $g(t)$ and $g_1(t)$ both delayed by T]. Both pulses in Fig. P3.3-2d and e can be expressed as $g(t-T)+g_1(t+T)$ [the sum of $g(t)$ delayed by T and $g_1(t)$ advanced by T] for some suitable choice of T. The pulse in Fig. P3.3-2f can be obtained by time-expanding $g(t)$ by a factor of 2 and then delaying the resulting pulse by 2 seconds [or by first delaying $g(t)$ by 1 second and then time-expanding by a factor of 2].

Figure P.3.3-2



3.3-3 Using only the time-shifting property and Table 3.1, find the Fourier transforms of the signals shown in Fig. P3.3-3.

Figure P.3.3-3



Hint: The signal in Fig. P3.3-3a is a sum of two shifted rectangular pulses. The signal in Fig. P3.3-3b is $\sin t\,[u(t) - u(t-\pi)] = \sin t\,u(t) - \sin t\,u(t-\pi) = \sin t\,u(t) + \sin(t-\pi)$

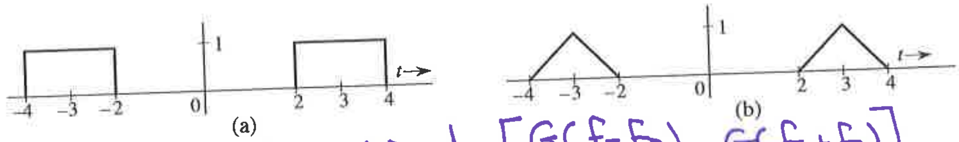
$u(t - \pi)$. The reader should verify that the addition of these two sinusoids indeed results in the pulse in Fig. P3.3-3b. In the same way, we can express the signal in Fig. P3.3-3c as $\cos t\, u(t) + \sin(t - \pi/2)u(t - \pi/2)$ (verify this by sketching these signals). The signal in Fig. P3.3-3d is $e^{-at}[u(t) - u(t - T)] = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t - T)$.

3.3-4 Use the time-shifting property to show that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t + T) + g(t - T) \Longleftrightarrow 2G(f)\cos 2\pi fT$$

This is the dual of Eq. (3.36). Use this result and pairs 17 and 19 in Table 3.1 to find the Fourier transforms of the signals shown in Fig. P3.3-4.

Figure P.3.3-4



Eq. 3.36 $g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f-f_0) - G(f+f_0)]$

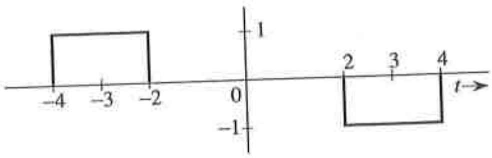
3.3-5 Prove the following results:

$$g(t)\sin 2\pi f_0 t \Longleftrightarrow \frac{1}{2j}[G(f - f_0) - G(f + f_0)]$$

$$\frac{1}{2j}[g(t + T) - g(t - T)] \Longleftrightarrow G(f)\sin 2\pi fT$$

Use the latter result and Table 3.1 to find the Fourier transform of the signal in Fig. P3.3-5.

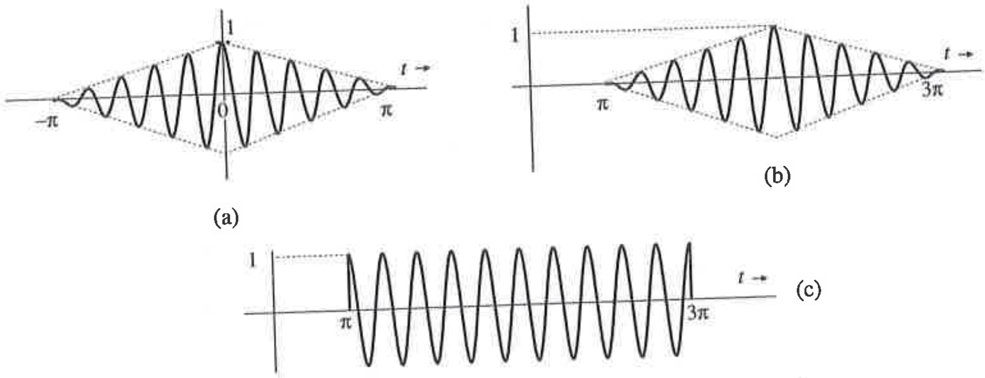
Figure P.3.3-5



3.3-6 The signals in Fig. P3.3-6 are modulated signals with carrier $\cos 10t$. Find the Fourier transforms of these signals by using the appropriate properties of the Fourier transform and Table 3.1. Sketch the amplitude and phase spectra for Fig. P3.3-6a and b.

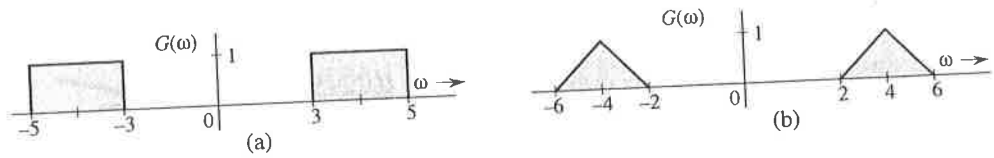
Hint: These functions can be expressed in the form $g(t)\cos 2\pi f_0 t$.

Figure P.3.3-6



3.3-7 Use the frequency shift property and Table 3.1 to find the inverse Fourier transform of the spectra shown in Fig. P3.3-7. Notice that this time, the Fourier transform is in the ω domain.

Figure P.3.3-7



3.3-8 A signal $g(t)$ is band-limited to B Hz. Show that the signal $g^n(t)$ is band-limited to nB Hz.

Hint: $g^2(t) \Longleftrightarrow [G(f) * G(f)]$, and so on. Use the width property of convolution.

3.3-9 Find the Fourier transform of the signal in Fig. P3.3-3a by three different methods:

(a) By direct integration using the definition (3.9a).

(b) Using only pair 17 Table 3.1 and the time-shifting property.

(c) Using the time differentiation and time-shifting properties, along with the fact that $\delta(t) \Longleftrightarrow 1$.

Hint: $1 - \cos 2x = 2\sin^2 x$.

3.3-10 The process of recovering a signal $g(t)$ from the modulated signal $g(t)\cos 2\pi f_0 t$ is called **demodulation**. Show that the signal $g(t)\cos 2\pi f_0 t$ can be demodulated by multiplying it by $2\cos 2\pi f_0 t$ and passing the product through a low-pass filter of bandwidth B Hz [the bandwidth of $g(t)$]. Assume $B < f_0$. *Hint*: $2\cos^2 2\pi f_0 t = 1 + \cos 4\pi f_0 t$. Recognize that the spectrum of $g(t)\cos 4\pi f_0 t$ is centered at $2f_0$ and will be suppressed by a low-pass filter of bandwidth B Hz.

3.4-1 Signals $g_1(t) = 10^4\Pi(10^4 t)$ and $g_2(t) = \delta(t)$ are applied at the inputs of the ideal low-pass filters $H_1(f) = \Pi(f/20{,}000)$ and $H_2(f) = \Pi(f/10{,}000)$ (Fig. P3.4-1). The outputs $y_1(t)$ and $y_2(t)$ of these filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$.

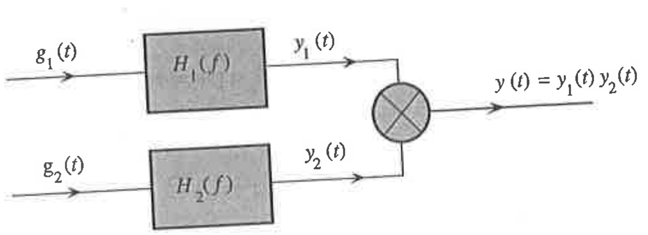
(a) Sketch $G_1(f)$ and $G_2(f)$.

(b) Sketch $H_1(f)$ and $H_2(f)$.

(c) Sketch $Y_1(f)$ and $Y_2(f)$.

(d) Find the bandwidths of $y_1(t)$, $y_2(t)$, and $y(t)$.

Figure P.3.4-1



3.5-1 For systems with the following impulse responses, which system is causal?

(a) $h(t) = e^{-at}u(t), \quad a > 0$

(b) $h(t) = e^{-a|t|}, \quad a > 0$

(c) $h(t) = e^{-a(t-t_0)}u(t - t_0), \quad a > 0$

(d) $h(t) = \text{sinc}(at), \quad a > 0$

(e) $h(t) = \text{sinc}[a(t - t_0)], \quad a > 0.$

3.5-2 Consider a filter with the transfer function

$$H(f) = e^{-k(2\pi kf)^2 - j2\pi f t_0}$$

Show that this filter is physically unrealizable by using the time domain criterion [noncausal $h(t)$] and the frequency domain (Paley-Wiener) criterion. Can this filter be made approximately realizable by choosing a sufficiently large t_0? Use your own (reasonable) criterion of approximate realizability to determine t_0.

Hint: Use pair 22 in Table 3.1.

3.5-3 Show that a filter with transfer function

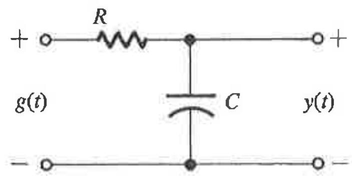
$$H(f) = \frac{2(10^5)}{(2\pi f)^2 + 10^{10}} e^{-j2\pi f t_0}$$

is unrealizable. Can this filter be made approximately realizable by choosing a sufficiently large t_0? Use your own (reasonable) criterion of approximate realizability to determine t_0.

Hint: Show that the impulse response is noncausal.

3.5-4 Determine the maximum bandwidth of a signal that can be transmitted through the low-pass RC filter in Fig. P3.5-4 with $R = 1000$ and $C = 10^{-9}$ if, over this bandwidth, the amplitude response (gain) variation is to be within 5% and the time delay variation is to be within 2%.

Figure P.3.5-4



3.5-5 A bandpass signal $g(t)$ of bandwidth $B = 2000$ Hz centered at $f = 10^5$ Hz is passed through the RC filter in Fig. P3.5-4 with $RC = 10^{-3}$. If over the passband, a variation of less than 2% in amplitude response and less than 1% in time delay is considered distortionless transmission, would $g(t)$ be transmitted without distortion? Find the approximate expression for the output signal.

3.6-1 A certain channel has ideal amplitude, but nonideal phase response (Fig. P3.6-1), given by

$$|H(f)| = 1$$
$$\theta_h(f) = -2\pi f t_0 - k \sin 2\pi f T \qquad k \ll 1$$

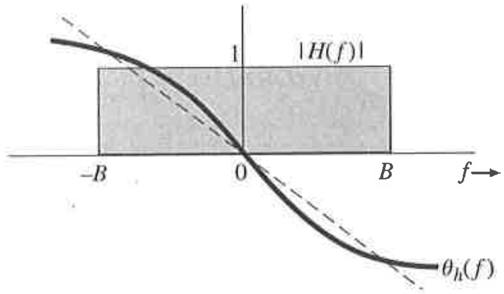
(a) Show that $y(t)$, the channel response to an input pulse $g(t)$ band-limited to B Hz, is

$$y(t) = g(t - t_0) + \frac{k}{2}[g(t - t_0 - T) - g(t - t_0 + T)]$$

Hint: Use $e^{-jk \sin 2\pi f T} \approx 1 - jk \sin 2\pi f T$.

(b) Discuss how this channel will affect TDM and FDM systems from the viewpoint of interference among the multiplexed signals.

Figure P.3.6-1

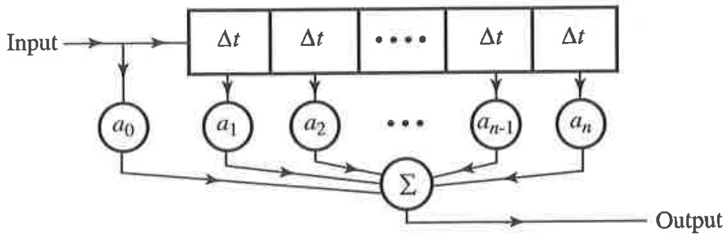


3.6-2 The distortion caused by multipath transmission can be partly corrected by a tapped delay-line equalizer. Show that if $\alpha \ll 1$, the distortion in the multipath system in Fig. 3.31a can be approximately corrected if the received signal in Fig. 3.31a is passed through the tapped delay-line equalizer shown in Fig. P3.6-2.

Hint: From Eq. (3.64a), it is clear that the equalizer filter transfer function should be $H_{\text{eq}}(f) = 1/(1+\alpha\, e^{-j2\pi f \Delta t})$. Use the fact that $1/(1-x) = 1+x+x^2+x^3+\cdots$ if $x \ll 1$ to show what should be the tap parameters a_i to make the resulting transfer function

$$H(f)H_{\text{eq}}(f) \approx e^{-j2\pi f t_d}$$

Figure P.3.6-2



3.7-1 Show that the energy of the Gaussian pulse

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

from direct integration is $1/2\sigma\sqrt{\pi}$. Verify this result by using Parseval's theorem to derive the energy E_g from $G(f)$. *Hint:* See pair 22 in Table 3.1. Use the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \pi \Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

3.7-2 Show that

$$\int_{-\infty}^{\infty} \text{sinc}^2\,(kt)dt = \frac{\pi}{4}$$